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QUANTUM TIME EVOLUTION OF A REPARAMETRIZATION INVARIANT SYSTEM

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ABSTRACT

In this paper we have discussed the problem of quantum time evolution for a reparametrized invariant system. The *problem of time* in quantum gravity is well known. This is due to the different nature of time in quantum mechanics and in general relativity. It mainly appears in the program of canonical quantization of general relativity. We will analyse this problem specifically for our proposed model and discuss the result for further study.

Keywords: General relativity; Quantum gravity; Problem of time; Deparametrization.

1 INTRODUCTION

Quantum mechanics [1] treats time as an absolute external parameter while in general relativity [2] time is merely a coordinate and not absolute. These mutually contradictory attributes of a single entity called *time* are the primary cause for the existence of the problem of time [3] in quantization program of gravity. In the canonical formulation of general relativity the spacetime metric, $g_{\mu\nu}$, is used as the fundamental variable of the theory. When quantization procedure is applied, we get the Wheeler-DeWitt equation [4]. This equation shows that the total Hamiltonian is constrained to vanish. The total Hamiltonian in this case is the sum of Hamiltonian and diffeomorphism constraints [5]. These constraints are in the core of the problem of time in quantum gravity. When quantized the corresponding quantum Hamiltonian operator becomes zero, which according to Dirac's proposal [3] simply means that the quantum state does not evolve with respect to any external evolution parameter. In other words, the Wheeler-DeWitt equation describes a state which is 'frozen' in time. In this paper we will discuss the problem of quantum time evolution in reparametrization invariant system [5]. Since the theory of general relativity is also reparametrization invariant in itself, this study will be helpful to provide some insight to the solution of the problem of time in such systems.

2 TOTAL HAMILTONIAN AND CONSTRAINT

We write Jacobi action [6,7] for our system of two coupled particles of masses m and M as

$$S(\xi,\eta) = \int d\tau \sqrt{\left(\frac{m}{2}\dot{\xi}^2 + \frac{M}{2}\dot{\eta}^2\right)\left(E - \frac{1}{2}k\left(\xi^2 + \eta^2\right)\right)} \,, \tag{1}$$

where $\dot{\xi} = \frac{d\xi}{d\tau}$ and $\dot{\eta} = \frac{d\eta}{d\tau}$, and τ is the proper time parameter. E can be some energy function. It can be proved that this action is reparametrization invariant. This reparametrization invariance ensures the condition that the actual physical trajectories of the particle or the system are independent of the parameter used to parametrize the path followed. We should also note that appearance of the square root in eq. (1), in fact in the Lagrangian of the action, is a sign that the action $S(\xi, \eta)$ is related to the motion of relativistic particles.

The canonical Hamiltonian corresponding to the Lagrangian of this action is given by

$$H = p_\xi \dot{\xi} + p_\eta \dot{\eta} - L \,, \qquad (2)$$

where L is the Lagrangian of the action $S(\xi, \eta)$, and p_ξ, p_η are canonical momenta conjugate to ξ, η respectively. It is the property of reparametrization invariant action that corresponding canonical Hamiltonian is constrained to vanish. So, the constraint equation is

$$H = 0. \qquad (3)$$

In Hamiltonian formulation the constraint given by equation (3) is called a primary constraint.

The total Hamiltonian for the system is given as

$$H_T = \lambda\left[\frac{2(p_\xi)^2}{m} + \frac{2(p_\eta)^2}{M} + \frac{k}{2}\left(\xi^2 + \eta^2\right) - E\right], \qquad (4)$$

where λ is the Lagrangian multiplier and k is same as appears in equation (1). This constrained situation leads to time-independent Schrödinger equation. It simply means that there is no evolution at all hence problem of time.

3 NEW DYNAMICS

Suppose now that the variable η is a monotonic evolution parameter with respect to which rest of the variables evolve, i.e.

$$\frac{d\eta}{d\tau} > 0 \quad . \qquad (5)$$

The new action will then be

$$S_n = \int d\eta \left(\frac{m}{2}\left(\frac{d\xi}{d\eta}\right)^2 + \frac{M}{2}\right)\left(E - \frac{k}{2}\left(\xi^2 + \eta^2\right)\right) \quad . \qquad (6)$$

This action describes the evolution of ξ with respect to η; in this situation η is independent variable while ξ is dependent variable. Though the system is same but the dynamics is now described by its evolution with respect to parameter η. One can see that the action no longer explicitly depends upon the arbitrary parameter τ. The Hamiltonian corresponding to action (6) will be

$$\acute{H} = \frac{1}{\sqrt{m}}\sqrt{\frac{M}{2\left(m\left(E - {}^1\!/_2 k\left(\xi^2 + \eta^2\right)\right) - 2(p_\xi)^2\right)}}\left(2(p_\xi)^2 - m\left(E - {}^1\!/_2\, k\left(\xi^2 + \eta^2\right)\right)\right) \qquad (7)$$

4 THE QUANTUM EVOLUTION

Our system is now gauge fixed whose canonical Hamiltonian is given by equation (7). In our case now η plays the role of evolution parameter or time and ξ is the coordinate affixed to the evolving system. Therefore, the phase space variables are ξ and p_ξ and evolution can be seen with respect to η.

We can now write the time dependent Schrödinger equation for our system as

$$i\hbar \frac{\partial \psi(\xi, \eta)}{\partial \eta} = \widehat{H} \psi(\xi, \eta), \qquad (8)$$

where $\widehat{H}$ is the Hamiltonian operator obtained from equation (7) by replacing classical variables with corresponding quantum mechanical operators. Obviously, we have attained a sort of deparametrization of our theory.

5 CONCLUSION

We have discussed the problem of time for our proposed model. We considered action which is reparametrization invariant. It has been shown that a viable option to get around this problem of time is to consider one of the degrees of freedom of the system as the evolution parameter. This gauge fixing allows the system to evolve with respect this chosen degree of freedom. Then one can obtain the quantum mechanical time dependent Schrödinger equation for the system. However, we also see several challenges. The major challenge is the appearance of square root Hamiltonian operator. The solution to this situation is to square the Hamiltonian. But this exercise introduces spurious solutions. We also observe the explicit dependence of $\acute{H}$ on η, this is a serious issue because it shows that the system under consideration is not time translation invariant. In other words, the energy associated with η is not conserved. Moreover, the Schrödinger equation (8) will be highly complicated to solve. In conclusion we can say that using a deparametrized Hamiltonian we can formally write a quantum evolution equation which under some approximation can be used to get some insight for quantum dynamics of our system.

REFERENCES

[1] Bohm A. Quantum mechanics: Foundations and applications. New Delhi: Springer (India) Private Limited; 2003.

[2] Wald R M. General relativity. Chicago: University of Chicago Press;1984.

[3] Bojowald M. Canonical gravity and applications. Cambridge: Cambridge University Press; 2011.

[4] DeWitt B S. Quantum theory of gravity. I. The canonical theory. Phys. Rev. 1967 August; **160**: p. 1113-1148.

[5] Kiefer C. Quantum gravity.2nd ed. Oxford: Oxford University Press; 2004.

[6] Goldstein H, Poole C, Safko J. Classical mechanics. New Delhi: Dorling Kindersley (India) Private Limited; 2009.

[7] Frauca Á M. Reassessing the problem of time of quantum gravity. Gen. Relativ. Gravit. 2023; **55: 21**. p. 1-37.