



(RESEARCH ARTICLE)



Analytical solutions of fractional Poisson's Equation with Riesz Derivatives

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International Journal of Science and Research Archive, 2025, 16(01), 1595-1600

Publication history: Received on 14 June 2025; revised on 20 July 2025; accepted on 22 July 2025

Article DOI: <https://doi.org/10.30574/ijrsra.2025.16.1.2197>

Abstract

This paper provides an analytical framework for solving the fractional Poisson's equation involving Riesz fractional derivatives of order α in a fractional dimensional space of dimension D . By employing the Fourier transform technique, we derive explicit solutions and establish a fundamental link between the order of fractional differentiation and the dimension of the space. A generalized Gauss's law is formulated for fractional spaces, and the total electric flux is expressed as a function of α and D . Furthermore, a fractional multipole expansion using Gegenbauer polynomials is introduced, enabling a compact representation of higher-order terms in fractional space. These results offer a significant extension of classical electromagnetic theory to fractional dimensions, providing a basis for modeling complex systems with anisotropic or confined geometries. The developed approach also opens the possibility of extending the analysis to fractional Helmholtz equations in future research.

Keywords: Fractional Calculus; Riesz Fractional Derivative; Fractional Dimensional Space; Fractional Poisson's Equation; Fourier Transform Method

1. Introduction

Fractional variational calculus is an extension of the classical calculus of variations, incorporating fractional derivatives and integrals to address problems involving memory and hereditary effects. Over the past few decades, this field has gained significant attention across multiple disciplines, including mathematics, physics, mechanics, economics, and engineering, due to its ability to model complex and anomalous phenomena. Fractional operators provide a more accurate description of processes with non-local dynamics and fractal characteristics, making them indispensable for modern applications [1-8]. Recent research has explored fractional Euler-Lagrange equations, free-boundary problems, isoperimetric constraints, and generalized natural boundary conditions within the context of Riesz-Caputo derivatives. These developments demonstrate the growing interest in extending classical optimality conditions to fractional domains, thereby enabling new formulations in optimal control, fractional mechanics, and other applied sciences [5-7].

Pathak and Somvanshi thoroughly studied the existence of solutions of fractional differential equations by using fixed point theorems, which is important in nonlinear dynamics [9-13].

The objective of this paper is to establish necessary optimality conditions for a broad class of fractional variational problems involving the Riesz-Caputo derivative. We begin by recalling fundamental definitions and properties of fractional operators required for subsequent analysis. We then derive generalized Euler-Lagrange equations when the interval of integration differs from the domain of the fractional derivative, followed by the treatment of integral constraints through isoperimetric problems. The results obtained in this work contribute to the theoretical foundation of fractional variational calculus and open pathways for further research in modeling systems with non-local and memory-dependent dynamics.

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2. Fractional Poisson's Equation

One of the effective methods to find the solution of the electrostatic problems is to find the solutions of Poisson's equation and then one can find the electric field for the Source Charge. The starting point is to consider the fractional Poisson's equation as follows.

$$(-\Delta)^{\alpha/2} \phi(\mathbf{r}) = \frac{\rho}{\epsilon_0}, \quad 1 < \alpha \leq 2, \quad \dots\dots\dots (1)$$

Where ρ is the source electric charge density, \vec{r} is the D dimensional vector $\Delta = \frac{\partial^2}{\partial r^2}$ is the Laplacian and the operator $(-\Delta)^{\alpha/2}$ is the α dimensional generalization of fractional quantum Riesz derivative [2,4,30,31]

$$(-\Delta)^{\alpha/2} \psi(\mathbf{r}, t) = \left(\frac{1}{2\pi}\right)^D \int_{K^D} e^{i\mathbf{k}\cdot\mathbf{r}} e^{i\mathbf{k}\cdot\mathbf{r}} e^{-ik.r} d^D r. \quad \dots\dots\dots (2)$$

To solve (1), we use the Fourier transforms as

$$F\phi(\mathbf{r}) = g(\mathbf{k}) = \int_{R^D} \phi(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} d^D r. \quad \dots\dots\dots (3)$$

$$F\rho(\mathbf{r}) = f(\mathbf{k}) = \int_{R^D} \rho(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}} d^D r. \quad \dots\dots\dots (4)$$

The inverse Fourier transform reads as

$$F^{-1}\phi(\mathbf{r}) = \left(\frac{1}{2\pi}\right)^D \int_{K^D} g(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}} d^D k. \quad \dots\dots\dots (5)$$

$$F^{-1}\rho(\mathbf{r}) = \left(\frac{1}{2\pi}\right)^D \int_{K^D} f(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}} d^D k. \quad \dots\dots\dots (6)$$

Applying the Fourier transform with respect to spatial variable x we have

$$g^{(k)} = \frac{f(k)}{k^\alpha} \quad \dots\dots\dots (7)$$

Hence, we obtain the solutions $\phi(\mathbf{r})$ as

$$\phi(\mathbf{r}) = \int_{R^D} G(\mathbf{r}-\mathbf{r}') \frac{\rho(\mathbf{r}')}{\epsilon_0} d^D r'. \quad \dots\dots\dots (8)$$

Where, $G(\mathbf{r}-\mathbf{r}')$ is the Green's function (kernel) and is given by

$$G(\mathbf{r}-\mathbf{r}') = \left(\frac{1}{2\pi}\right)^D \int_{K^D} \frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{k^\alpha} d^D \mathbf{k}. \quad \dots\dots\dots (9)$$

Now using the following transformation defined in [2,4]

$$\int_{K^D} \varphi(k) e^{ik \cdot (r-r')} d^D k = \int_0^\alpha \varphi(\rho) \rho^{D-1} d\rho \int_{S_{D-1}} e^{i\rho(r-r') \cdot \sigma} d\sigma \dots\dots\dots (10)$$

We obtain

$$\int_{K^D} \frac{e^{ik \cdot (r-r')}}{k^\alpha} d^D k = \int_0^\alpha \rho^{D-\alpha-1} d\rho \int_{S_{D-1}} e^{i\rho(r-r') \cdot \sigma} d\sigma \dots\dots\dots (11)$$

Nothing that

$$\int_{S_{D-1}} f(r - \sigma) d\sigma = \frac{2\pi^{(D-1)/2}}{\Gamma\left(\frac{D-1}{2}\right)} \int_{-1}^1 f(|x|t) (1-t^2)^{(D-3)/2} dt \dots\dots\dots (12)$$

And also

$$J_\nu(x) = \frac{(x/2)^\nu}{\Gamma(\nu+1/2)\sqrt{\pi}} \int_{-1}^1 e^{ixt} (1-t^2)^{(\nu-1/2)} dt \dots\dots\dots (13)$$

Where $J_\nu(x)$ is the Bessel function of the first kind. we arrive at

$$\int_{S_{D-1}} e^{ix \cdot \sigma} d\sigma = \frac{2\pi^{D/2}}{|x|^{D/2-1}} J_{d/2-1}(|x|). \dots\dots\dots (14)$$

The Green's Function (9) takes the form

$$G(r - r') = (2\pi)^{-D/2} \int_0^\alpha \frac{\rho^{D-\alpha-1} J_{D/2}(\rho |r - r'|)}{(\rho |r - r'|)^{D/2-1}} d\rho \dots\dots\dots (15)$$

Using the Mellin transform of the Bessel function [2,32]

$$\int_0^\alpha \rho^\beta J_\nu(\rho) d\rho = \frac{2^\beta \Gamma\left(\frac{\nu+\beta+1}{2}\right)}{\Gamma\left(\frac{\beta-\nu+1}{2}\right)} \dots\dots\dots (16)$$

We obtain

$$G(r - r') = \frac{1}{2^\alpha \pi^{D/2}} \frac{\Gamma\left(\frac{D-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} \frac{1}{|r - r'|^{D-\alpha}} \dots\dots\dots (17)$$

The solution of Poission's equation (1) is given by

$$\phi(r)_{\alpha,D} = k_{\alpha,D} a \int \frac{\rho(r') d^D r'}{|r - r'|^{D-\alpha}} \dots\dots\dots (18)$$

Where the constant k_α, D is defined as

$$k_{\alpha,D} = \frac{1}{2^\alpha \pi^{D/2} \epsilon_0} \frac{\Gamma(\frac{D-\alpha}{2})}{\Gamma(\frac{\alpha}{2})} \dots\dots\dots (19)$$

For $\alpha=2$ and $D=3$, We have $k_{2,3} = \frac{1}{4\pi\epsilon_0}$

For different value of α and D , we obtain the potential for any D fractional dimensional space and for any order Riesz fractional derivative of order α via the relation

$$(-\Delta)^{\alpha/2} \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|^{D-\alpha}} \right) = \frac{2^\alpha \pi^{D/2} \Gamma(\frac{\alpha}{2})}{\Gamma(\frac{D-\alpha}{2})} \delta^D(\mathbf{r}-\mathbf{r}') \dots\dots\dots (20)$$

Where $\delta^D(\mathbf{r}-\mathbf{r}')$ is the D dimensional fractional Dirac delta function. For $\alpha=2$ and $D=3$ we have.

$$\Delta - \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) = -4\pi\delta^3(\mathbf{r}-\mathbf{r}') \dots\dots\dots (21)$$

3. Gauss's Law

Gauss's Law gives the relation between electric field and the charge enclosed in a closed Gauss surface. To derive Gauss's law in D dimension fractional space, let us consider a closed D dimensional sphere of radius R with its center at the origin of the coordinate system. The total flux of the electric field on the surface of the closed sphere is

$$\oint \vec{E} \cdot \vec{dA} = \int_0^\pi \left(-\frac{\partial\phi(r)}{\partial r} \right) r^{D-1} (\sin\theta)^{D-2} d\theta = \frac{qr^{\alpha-2}}{2^{\alpha-1} \epsilon_0} \frac{\Gamma(\frac{D-\alpha}{2})(D-2)}{\Gamma(\frac{\alpha}{2})} \dots\dots\dots (22)$$

For $\alpha=2$ and $D=3$

$$\oint \vec{E} \cdot \vec{dA} = \frac{q}{\epsilon_0} \dots\dots\dots (23)$$

4. Fractional Multipole Expansion of Riesz Potential of Order α in Fractional D Dimensional Space

Multipole, expansion of sources is a very well-known subject in electromagnetism, and has been studied extensively (e.g. [133-35] In this section we will obtain the multipole expansion in fractional space. The potential (18) can be expanded using the definition of generating function of Gegenbauer polynomials and considering that $r > r'$

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|^{D-\alpha}} = \sum_{l=0}^{\infty} \frac{c_l^{(D-\alpha)/2}(\cos\vartheta) r'^l}{r^{D-\alpha+1}} \cdot r > r' \dots\dots\dots (24)$$

Where $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}' = \cos\vartheta$ and $C_l^{(D-\alpha)/2}(\cos\vartheta)$ are the Gegenbauer polynomials in $\cos\vartheta$ and the forms of the first few Gegenbauer polynomials are given by

$$C_0^{(D-\alpha)/2}(x) = 1 \dots\dots\dots (25)$$

$$C_1^{(D-\alpha)/2}(x) = (D-\alpha)x, \dots\dots\dots (26)$$

$$C_2^{(D-\alpha)/2}(x)a = \left(\frac{D-\alpha}{2}\right)((D-\alpha+2)x^2-1) \dots\dots\dots (27)$$

The potential (18), may be expressed as

$$\phi(r)_{\alpha,D} = k_{\alpha,D} \sum_{i=0}^{\infty} \frac{q_1^{D,\alpha}}{r^{D-\alpha+1}}, \dots\dots\dots (28)$$

Where $q_1^{D,\alpha}$ are the fractional multipole terms of order i and are given by

$$q_1^{D,\alpha} = \int \rho(r') r'^l C_1^{(D-\alpha)/2}(\cos \vartheta) d^D r'. \dots\dots\dots (29)$$

For l=0 we have only fractional monopole term as

$$\phi(r)_{\alpha,D} = k_{\alpha,D} \frac{q}{r^{D-\alpha}} \dots\dots\dots (30)$$

Where q represents the monopole charge.

5. Conclusion

In this paper we have given an application of fractional calculus in electromagnetic theory and introduce a solution of the fractional Poisson's equation with Riesz derivative. The Fourier transform method is used to solve this equation and it is observed that fractional derivative and the fractional dimensional space are connected simultaneously via relation (20), which means that fractionality in the derivatives is due to the fractionality in the space. Another interesting result we have obtained is the new definition of the constant $k\alpha, D$.

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