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On best proximity points in b-multiplicative metric spaces using R-functions with an application

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Abstract

In this paper, we introduce some rational cyclic conditions in b-multiplicative metric spaces. These conditions generalize existing rational cyclic conditions studied in standard metric frameworks. We establish several best proximity point theorems for a pair of mappings that satisfy these conditions in b-multiplicative metric spaces, thereby broadening the scope of fixed point and best proximity point theory. To illustrate the theoretical results, we provide an application to a nonlinear integral equation.

Keywords: b-Multiplicative Metric Space; R -Function; Rational Cyclic Condition; Cyclic Map; Best Proximity Point

1. Introduction

The concept of b-metric spaces, introduced by Bakhtin in 1989, serves as a natural extension of traditional metric spaces and has since inspired a wealth of research, particularly in the area of fixed-point theory. Building upon this foundation, various generalizations of the Banach contraction principle have been proposed to accommodate the broader structure of b-metric spaces.

In classical analysis, the Banach contraction principle asserts that every contraction mapping on a complete metric space has a unique fixed point. This fundamental result has been extended in numerous directions, including settings where the mappings are not necessarily self-maps. When dealing with non-self mappings between two subsets A and B of a metric space, the fixed point equation $Tx = x$ may not have a solution.

The concept of a best proximity point emerges from the goal of minimizing the error $d(x, Tx)$, especially under the condition $d(A, B) = \inf\{d(x, y): x \in A, y \in B\}$. A best proximity point is defined as a point x satisfying $d(x, Tx) = d(A, B)$ thereby representing an optimal approximate solution are called best proximity points of the mapping T.

Cyclic and multivalued mappings have been widely studied in this context. Notably, Kirk and others introduced various forms of cyclic mappings and explored their connections to best proximity points. Further developments by researchers such as Eldered, Veeramani, and Sadiq Basha introduced classes like cyclic contractions, K-cyclic and C-cyclic mappings, and a range of generalized contractive conditions involving auxiliary functions [See 6-7,12]

Motivated by these developments, recent studies have extended best proximity point theorems to settings such as S-metric, G-metric, and other generalized spaces[See 11,13,14,18]. In particular, conditions like weak R −K, R− C, and combinations thereof have been shown to yield meaningful proximity results for various mapping pairs. Building upon the foundational work of Ghezellou et.al [9], who studied fixed point theorems under rational cyclic conditions in b-

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metric spaces, we extend these results to the framework of b-multiplicative metric spaces by relaxing certain contractive conditions.

In this article, we aim to extend these ideas to a newly generalized framework called the b-multiplicative metric space ($\mathfrak{b}-\mathcal{MMS}$), which blends the features of b-metric and multiplicative metric spaces. We introduce novel cyclic-type contractive conditions in this setting—namely, weak R–K rational cyclic($\mathcal{RK}\mathfrak{RC}$), weak R-C rational cyclic($\mathcal{RC}\mathfrak{RC}$), and the combined weak R–KC rational cyclic($\mathcal{RKC}\mathfrak{RC}$) conditions. Under these assumptions, we establish new best proximity point theorems for a pair of mappings that satisfy the proposed conditions within $\mathfrak{b}-\mathcal{MMS}$.

2. Preliminaries

2.1. Definition [2]

Let χ be a non-empty set and $\mu \geq 1$ be a given real number. A function $d^*: \chi \times \chi \to [1, \infty)$ is called a b −multiplicative metric with coefficient μ if the following conditions hold

- $d^*(\mathfrak{p}, \mathfrak{q}) > 1$ for all $\mathfrak{p}, \mathfrak{q} \in \chi$ with $\mathfrak{p} \neq \mathfrak{q}$ and $d^*(\mathfrak{p}, \mathfrak{q}) = 1$ iff $\mathfrak{p} = \mathfrak{q}$
- $d^*(\mathfrak{p}, \mathfrak{q}) = d^*(\mathfrak{q}, \mathfrak{p})$ for all $\mathfrak{p}, \mathfrak{q} \in \chi$
- $d^*(\mathfrak{p}, \mathfrak{r}) \leq [d^*(\mathfrak{p}, \mathfrak{q}). d^*(\mathfrak{q}, \mathfrak{r})]^\mu$ for all $\mathfrak{p}, \mathfrak{q}, \mathfrak{r} \in \chi$

The triplet $(\chi,\ d^*, \mu)$ is called $\mathfrak{b}-\mathcal{MMS}$.

2.2. Definition [2]

Let (χ, d^*, μ) be a $\mathfrak{b}-\mathcal{MMS}$.

- A sequence $\{\mathfrak{p}_n\}$ is $\mathfrak{b}$-multiplicative convergent iff $\exists\ \mathfrak{p} \in \chi$ such that

$d^*(\mathfrak{p}_n, \mathfrak{p}) \to 1$ as $n \to \infty$.

- A sequence $\{\mathfrak{p}_n\}$ is called $\mathfrak{b}$-multiplicative Cauchy iff

$d^*(\mathfrak{p}_m, \mathfrak{p}_n) \to 1$ as $n, m \to \infty$.

- A $\mathfrak{b}-\mathcal{MMS}$ (χ, d^*, μ) is said to be complete if every $\mathfrak{b}$-multiplicative Cauchy sequence in χ is $\mathfrak{b}$ - convergent to a point in χ.

2.3. Definition [15]

Let A and B are nonempty subset of a nonempty set X, a mapping $T: A \cup B \to A \cup B$, is said to be cyclic if it satisfies the conditions T(A) ⊂ B and T(B) ⊂ A. A point x ∈ A ∪ B is referred to as a best proximity point of T if it holds that d(x, T x) = d(A, B),

where d(A, B) = inf{d(x, y): x ∈ A, y ∈ B}.

2.4. Definition [19]

Let A and B be non-empty subset of a multiplicative metric space (X, d^*). A mapping $T: A \cup B \to A \cup B$ is called multiplicative cyclic contraction if $\exists\ \lambda \in [0,1) \ni d^*(Tx, Ty) \leq d^*(x, y)^\lambda \cdot d^*(\mathrm{A}, \mathrm{B})^{1-\lambda}, \forall\ x \in A\ and\ y \in B$.

2.5. Definition [5]

A function $\varphi: [0, \infty) \to [0,1)$ is called an R-functions (or $\mathcal{MT}-$functions) if for every $t \in [0, \infty)$, it holds that $\lim_{s \to t^+} \sup \varphi(s) < 1$.

2.6. Definition [9]

Let A and B be non-empty subsets of a $\mathfrak{b}$-metric space (χ, d, μ) and $\zeta: A \to B$ and $\Gamma: B \to A$ be two mappings. The pair of maps $\zeta\ and\ \Gamma$ is claimed to satisfy

- Weak $\mathfrak{b} - \mathcal{RK}\mathfrak{R}\mathcal{C}$ condition if ∃ an R function φ ∋

$$d(\zeta\mathfrak{u}, \Gamma\mathfrak{v}) \leq \frac{1}{2\mu}\varphi\big(d(\mathfrak{u},\mathfrak{v})\big)\left(\frac{d(\mathfrak{u},\zeta\mathfrak{u}).d(\mathfrak{u},\Gamma\mathfrak{v})}{d(\mathfrak{u},\zeta\mathfrak{u}).d(\mathfrak{v},\zeta\mathfrak{v})}\right) + \frac{1}{\mu}(1-\varphi\big(d(\mathfrak{u},\mathfrak{v})\big)d(A,B)$$

$\forall\ \mathfrak{u} \in A, \mathfrak{v} \in B$ and $\mathfrak{u} \neq \Gamma\mathfrak{v}$

- Weak $\mathfrak{b} - \mathcal{RC}\mathfrak{R}\mathcal{C}$ condition if ∃ an R function φ ∋

$$d(\zeta\mathfrak{u}, \Gamma\mathfrak{v}) \leq \frac{1}{2\mu}\varphi\big(d(\mathfrak{u},\mathfrak{v})\big)\left(\frac{d(\mathfrak{u},\zeta\mathfrak{u}).d(\mathfrak{u},\Gamma\mathfrak{v})}{d(\mathfrak{u},\Gamma\mathfrak{v}).d(\mathfrak{v},\zeta\mathfrak{u})}\right) + \frac{1}{\mu}(1-\varphi(d(\mathfrak{u},\mathfrak{v}))\ d(A,B)$$

$\forall\ \mathfrak{u} \in A, \mathfrak{v} \in B$ and $\mathfrak{u} \neq \Gamma\mathfrak{v}$

- Weak $\mathfrak{b} - \mathcal{RKC}\mathfrak{R}\mathcal{C}$ condition if ∃ a pair of R function φ, ψ ∋

$$d(\zeta\mathfrak{u}, \Gamma\mathfrak{v}) \leq \frac{\mathfrak{x}_1}{\mu}\varphi\big(d(\mathfrak{u},\mathfrak{v})\big)\left(\frac{d(\mathfrak{u},\zeta\mathfrak{u}).d(\mathfrak{u},\Gamma\mathfrak{v})}{d(\mathfrak{u},\Gamma\mathfrak{u}).d(\mathfrak{v},\zeta\mathfrak{v})}\right) + \frac{\mathfrak{x}_2}{\mu}\psi\big(d(\mathfrak{u},\mathfrak{v})\big)\left(\frac{d(\mathfrak{u},\zeta\mathfrak{u}).d(\mathfrak{u},\Gamma\mathfrak{v})}{d(\mathfrak{u},\Gamma\mathfrak{v}).d(\mathfrak{v},\zeta\mathfrak{u})}\right) + \frac{1}{\mu}(1 - 2\big[\mathfrak{x}_1\varphi\big(d(\mathfrak{u},\mathfrak{v}) + \mathfrak{x}_2\psi\big(d(\mathfrak{u},\mathfrak{v})\big)\big]\big)d(A,B)$$

$\forall\ \mathfrak{u} \in A, \mathfrak{v} \in B$ and $\mathfrak{u} \neq \Gamma\mathfrak{v}$ and $\mathfrak{x}_1, \mathfrak{x}_2 \leq \frac{1}{2}$

2.7. Theorem ([15])

Let : [0,∞) → [0, 1) be a given function. The following statements are equivalent.

- The function φ qualifies as an R -function.
- For every non increasing sequence $\{x_n\} \subseteq [0,1)$, it holds that $0 \leq \sup \varphi(x_n) < 1$.
- For any strictly decreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in the interval [0,1), the function φ acts as a contractive factor, one has $0 \leq sup_{n\in\mathbb{N}}\ \varphi(x_n) < 1$.

3. Main results

3.1. Definition

Let A and B be non-empty subsets of a $\mathfrak{b} - \mathcal{MMS}\ (\chi, d^*, \mu)$ and $\zeta: A \to B$ and $\Gamma: B \to A$ be two mappings. The pair (ζ, Γ) is said to fulfil the condition

- Weak $\mathfrak{b} - \mathcal{RK}\mathfrak{R}\mathcal{C}$ condition if ∃ an R function φ ∋

$$d^*(\zeta\mathfrak{u}, \Gamma\mathfrak{v}) \leq \left(\frac{d^*(\mathfrak{u},\zeta\mathfrak{u}).d^*(\mathfrak{u},\Gamma\mathfrak{v})}{d^*(\mathfrak{u},\zeta\mathfrak{u}).d^*(\mathfrak{v},\zeta\mathfrak{v})}\right)^{\frac{1}{2\mu}\varphi\left(d^*(\mathfrak{u},\mathfrak{v})\right)} .\, d(A,B)^{\frac{1}{\mu}(1-\varphi(d^*(\mathfrak{u},\mathfrak{v}))} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (3.1)$$

$\forall\ \mathfrak{u} \in A, \mathfrak{v} \in B$ and $\mathfrak{u} \neq \Gamma\mathfrak{v}$

- Weak $\mathfrak{b} - \mathcal{RC}\mathfrak{R}\mathcal{C}$ condition if ∃ an R function φ ∋

$$d^*(\zeta\mathfrak{u}, \Gamma\mathfrak{v}) \leq \left(\frac{d^*(\mathfrak{u},\zeta\mathfrak{u}).d^*(\mathfrak{u},\Gamma\mathfrak{v})}{d^*(\mathfrak{u},\Gamma\mathfrak{v}).d^*(\mathfrak{v},\zeta\mathfrak{u})}\right)^{\frac{1}{2\mu}\varphi\left(d^*(\mathfrak{u},\mathfrak{v})\right)} .\, d(A,B)^{\frac{1}{\mu}(1-\varphi(d^*(\mathfrak{u},\mathfrak{v}))} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (3.2)$$

$\forall\ \mathfrak{u} \in A, \mathfrak{v} \in B$ and $\mathfrak{u} \neq \Gamma\mathfrak{v}$

- Weak $\mathfrak{b} - \mathcal{RKC}\mathfrak{R}\mathcal{C}$ condition if ∃ a pair of R function φ, ψ ∋

$$d^*(\zeta\mathfrak{u}, \Gamma\mathfrak{v}) \leq \left(\frac{d^*(\mathfrak{u},\zeta\mathfrak{u}).d^*(\mathfrak{u},\Gamma\mathfrak{v})}{d^*(\mathfrak{u},\Gamma\mathfrak{u}).d^*(\mathfrak{v},\zeta\mathfrak{v})}\right)^{\frac{\mathfrak{x}_1}{\mu}\varphi\left(d^*(\mathfrak{u},\mathfrak{v})\right)} .\left(\frac{d^*(\mathfrak{u},\zeta\mathfrak{u}).d^*(\mathfrak{u},\Gamma\mathfrak{v})}{d^*(\mathfrak{u},\Gamma\mathfrak{v}).d^*(\mathfrak{v},\zeta\mathfrak{u})}\right)^{\frac{\mathfrak{x}_2}{\mu}\psi\left(d^*(\mathfrak{u},\mathfrak{v})\right)} .\, d(A,B)^{\frac{1}{\mu}(1-2[\mathfrak{x}_1\varphi\left(d^*(\mathfrak{u},\mathfrak{v})+\mathfrak{x}_2\psi\left(d^*(\mathfrak{u},\mathfrak{v})\right)\right)])}$$

$$\forall\ \mathfrak{u} \in A, \mathfrak{v} \in B \text{ and } \mathfrak{u} \neq \Gamma\mathfrak{v} \text{ and } \mathfrak{x}_1, \mathfrak{x}_2 \leq \frac{1}{2} \qquad \ldots\ldots\ldots\ldots \quad (3.3)$$

3.2. Theorem

Let A and B be non-empty subsets of a b −multiplicative metric spaces (χ, d^*, μ) and let the mappings $\zeta: A \to B$ and $\Gamma: B \to A$ be given. The pair (ζ, Γ) is said to satisfy the weak $b - R - KC$ multiplicative rational cyclic condition if ∃ a sequence $\{\mathfrak{u}_n\}$ in χ such that

$$\lim_{\Gamma\to\infty} d^*(\mathfrak{u}_n, \mathfrak{u}_{n+1}) = d^*(A, B)$$

Furthermore, the following statements are true:

(1) If the subsequence $\{\mathfrak{u}_{2n}\}_{n\in\mathbb{N}}$ has a convergent subsequence in A, then ∃ a point $\mathfrak{u} \in A, \ni d^*(\mathfrak{u}, \zeta\ \mathfrak{u}) = d^*(A, B)$
(2) If the subsequence $\{\mathfrak{u}_{2n-1}\}_{n\in N}$ has a convergent subsequence in B, then ∃ a point $\mathfrak{v} \in B, \ni d^*(\mathfrak{v}, \Gamma\mathfrak{v}) = d^*(A, B)$

Proof

Consider $\mathfrak{u}_0 \in A$, set $\mathfrak{u}_{2n+1} = \zeta\mathfrak{u}_{2n}$ and $\mathfrak{u}_{2n} = \Gamma\mathfrak{u}_{2n-1}\ \forall\ n \in N\ \cup \{0\}$. As $\zeta(A) \subseteq B$ and $\Gamma(B) \subseteq A$, we obtain $\{\mathfrak{u}_{2n}\}_{n\in N} \subset A$ and $\{\mathfrak{u}_{2n-1}\}_{n\in N} \subset B$.

Define $\varphi_0: \varphi(d^*(\mathfrak{u}_0, \mathfrak{u}_1))$ and $\psi_0 := \psi(d^*(\mathfrak{u}, \mathfrak{u}_1))$ by definition 3.1 we have

$$d^*(\mathfrak{u}_1, \mathfrak{u}_2) = d^*(\zeta\mathfrak{u}_0, \Gamma\mathfrak{u}_1) \leq \left(\frac{d^*(\mathfrak{u}_0, \mathfrak{u}_1).\, d^*(\mathfrak{u}_0, \mathfrak{u}_2)}{d^*(\mathfrak{u}_0, \mathfrak{u}_1).\, d^*(\mathfrak{u}_1, \mathfrak{u}_2)}\right)^{\frac{\mathfrak{x}_1}{\mu}\varphi_0} . \left(\frac{d^*(\mathfrak{u}_0, \mathfrak{u}_1).\, d^*(\mathfrak{u}_0, \mathfrak{u}_2)}{d^*(\mathfrak{u}_0, \mathfrak{u}_2).\, d^*(\mathfrak{u}_1, \mathfrak{u}_1)}\right)^{\frac{\mathfrak{x}_2}{\mu}\psi_0} . d^*(A, B)^{\frac{1}{\mu}(1-2(\mathfrak{x}_1\varphi_0+\mathfrak{x}_2\psi_0))}$$

$$\leq \left(\frac{d^*(\mathfrak{u}_0, \mathfrak{u}_1).\, d^*(\mathfrak{u}_0, \mathfrak{u}_2)}{d^*(\mathfrak{u}_0, \mathfrak{u}_2)^{\frac{1}{\mu}}}\right)^{\mathfrak{x}_1\varphi_0} . \left(\frac{d^*(\mathfrak{u}_0, \mathfrak{u}_1).\, d^*(\mathfrak{u}_0, \mathfrak{u}_2)}{d^*(\mathfrak{u}_0, \mathfrak{u}_2)}\right)^{\mathfrak{x}_2\psi_0} . d^*(A, B)^{(1-2(\mathfrak{x}_1\varphi_0+\mathfrak{x}_2\psi_0))}$$

$$\leq \left(d^*(\mathfrak{u}_0, \mathfrak{u}_1).\, (d^*(\mathfrak{u}_0, \mathfrak{u}_2))^{1-\frac{1}{\mu}}\right)^{\mathfrak{x}_1\varphi_0} .\ d^*(\mathfrak{u}_0, \mathfrak{u}_1)^{\mathfrak{x}_2\psi_0} . d^*(A, B)^{(1-2(\mathfrak{x}_1\varphi_0+\mathfrak{x}_2\psi_0))}$$

therefore,

$$d^*(\mathfrak{u}_1, \mathfrak{u}_2)^{(1-(\mathfrak{x}_1\varphi_0+\mathfrak{x}_2\psi_0))} \leq d^*(\mathfrak{u}_0, \mathfrak{u}_1)^{(\mathfrak{x}_1\varphi_0+\mathfrak{x}_2\psi_0)} . d^*(A, B)^{(1-2(\mathfrak{x}_1\varphi_0+\mathfrak{x}_2\psi_0))}$$

it follows that

$$d^*(\mathfrak{u}_1, \mathfrak{u}_2) \leq d^*(\mathfrak{u}_0, \mathfrak{u}_1)^{\frac{\mathfrak{x}_1\varphi_0+\mathfrak{x}_2\psi_0}{1-(\mathfrak{x}_1\varphi_0+\mathfrak{x}_2\psi_0)}} . d^*(A, B)^{\frac{1-2(\mathfrak{x}_1\varphi_0+\mathfrak{x}_2\psi_0)}{1-(\mathfrak{x}_1\varphi_0+\mathfrak{x}_2\psi_0)}}$$

we give the proof only for the case $\mathfrak{x}_1 \geq \mathfrak{x}_2$; the same reasoning applies to the case $\mathfrak{x}_1 \leq \mathfrak{x}_2$. Now if we suppose $\mathfrak{x}_1 \geq \mathfrak{x}_2$, we would have

$$d^*(\mathfrak{u}_1, \mathfrak{u}_2) \leq d^*(\mathfrak{u}_0, \mathfrak{u}_1)^{\frac{\varphi_0+\frac{\mathfrak{x}_2}{\mathfrak{x}_1}\psi_0}{\frac{1}{\mathfrak{x}_1}-(\varphi_0+\frac{\mathfrak{x}_2}{\mathfrak{x}_1}\psi_0)}} . d^*(A, B)^{\left(1-\frac{\varphi_0+\frac{\mathfrak{x}_2}{\mathfrak{x}_1}\psi_0}{\frac{1}{\mathfrak{x}_1}-(\varphi_0+\frac{\mathfrak{x}_2}{\mathfrak{x}_1}\psi_0)}\right)}$$

Accordingly

$$d^*(\mathfrak{u}_1, \mathfrak{u}_2) \leq d^*(\mathfrak{u}_0, \mathfrak{u}_1)^{\kappa_0} . d^*(A, B) . d^*(A, B)^{-\kappa_0}$$

$$\frac{d^*(\mathfrak{u}_1, \mathfrak{u}_2)}{d^*(A, B)} \leq \left\{\frac{d^*(\mathfrak{u}_0, \mathfrak{u}_1)}{d^*(A, B)}\right\}^{\kappa_0}$$

Where $\kappa_0 = \frac{\varphi_0+\frac{\mathfrak{x}_2}{\mathfrak{x}_1}\psi_0}{\frac{1}{\mathfrak{x}_1}-\left(\varphi_0+\frac{\mathfrak{x}_2}{\mathfrak{x}_1}\psi_0\right)}$

Applying Definition 3.1 once again, we obtain

$$\frac{d^*(\mathfrak{u}_2,\mathfrak{u}_3)}{d^*(A,B)} \le \left(\frac{d^*(\mathfrak{u}_0,\mathfrak{u}_1)}{d^*(A,B)}\right)^{\kappa_1}$$

By induction we have

$$\frac{d^*(\mathfrak{u}_n,\mathfrak{u}_{n+1})}{d^*(A,B)} \le \left(\frac{d^*(\mathfrak{u}_{n-1},\mathfrak{u}_n)}{d^*(A,B)}\right)^{\kappa_{n-1}} \qquad \ldots\ldots\ldots\ldots\ldots \quad (3.4)$$

Where

$$\kappa_{n-1} = \frac{\varphi_{n-1} + \frac{\mathfrak{x}_2}{\mathfrak{x}_1}\psi_{n-1}}{\frac{1}{\mathfrak{x}_1} - \left(\varphi_{n-1} + \frac{\mathfrak{x}_2}{\mathfrak{x}_1}\psi_{n-1}\right)}$$

Since $\varphi(\mathrm{t}) + \psi(\mathrm{t}) < 1, \mathfrak{x}_1, \mathfrak{x}_2 \le \frac{1}{2}$ and $\mathfrak{x}_1 \ge \mathfrak{x}_2$,

$\forall$ t $\in [0,\infty)$ we see that

$$\frac{1}{\mathfrak{x}_1} - \left(\varphi(\mathrm{t}) + \frac{\mathfrak{x}_2}{\mathfrak{x}_1}\psi(\mathrm{t})\right) > 1$$

Consequently

$$\frac{\varphi(\mathrm{t}) + \frac{\mathfrak{x}_2}{\mathfrak{x}_1}\psi(\mathrm{t})}{\frac{1}{\mathfrak{x}} - \left(\varphi(\mathrm{t}) + \frac{\mathfrak{x}_2}{\mathfrak{x}_1}\psi(\mathrm{t})\right)} < 1$$

$\forall$ t $\in [0,\infty)$ by (3.4) we thus get $\kappa_{n-1} < 1$, therefore

$$\frac{d^*(\mathfrak{u}_n,\mathfrak{u}_{n+1})}{d^*(A,B)} \le \left(\frac{d^*(\mathfrak{u}_{n-1},\mathfrak{u}_n)}{d^*(A,B)}\right)$$

This gives $d^*(\mathfrak{u}_n,\mathfrak{u}_{n+1}) < d^*(\mathfrak{u}_{n-1},\mathfrak{u}_n)\ \forall\ n$.

Thus, the sequence $\{d^*(\mathfrak{u}_n,\mathfrak{u}_{n+1})\}$ is a strictly decreasing. Given that φ and ψ satisfy the weak R-condition,

According to Theorem 2.7

$$0 \le \sup_{n\in\mathbb{N}}(\varphi_n + \psi_n) < 1$$

Pick $\delta = \sup_{n\in\mathbb{N}}(\varphi_n + \psi_n)$, so $0 \le \delta < 1$. Seeing that

$$\varphi_n + \frac{\mathfrak{x}_2}{\mathfrak{x}_1}\psi_n \le \varphi_n + \psi_n \le \delta$$

We have

$$\frac{1}{\mathfrak{x}_1} - \varphi_n - \frac{\mathfrak{x}_2}{\mathfrak{x}_1}\psi_n \ge \frac{1}{\mathfrak{x}_1} - \delta$$

And

$$\frac{\varphi_n + \frac{\mathfrak{x}_2}{\mathfrak{x}_1}\psi_n}{\frac{1}{\mathfrak{x}_1} - \varphi_n - \frac{\mathfrak{x}_2}{\mathfrak{x}_1}\psi_n} \leq \frac{\delta}{\frac{1}{\mathfrak{x}_1} - \delta}$$

For all $n \in \mathbb{N}$. So

$$0 \leq \sup_{n\in\mathbb{N}} \frac{\varphi_n + \frac{\mathfrak{x}_2}{\mathfrak{x}_1}\psi_n}{\frac{1}{\mathfrak{x}_1} - \varphi_n - \frac{\mathfrak{x}_2}{\mathfrak{x}_1}\psi_n} \leq \frac{\delta}{\frac{1}{\mathfrak{x}_1} - \delta} < 1$$

Let

$$\kappa: \sup_{n\in\mathbb{N}} \frac{\varphi_n + \frac{\mathfrak{x}_2}{\mathfrak{x}_1}\psi_n}{\frac{1}{\mathfrak{x}_1} - \varphi_n - \frac{\mathfrak{x}_2}{\mathfrak{x}_1}\psi_n} \leq \frac{\delta}{\frac{1}{\mathfrak{x}_1} - \delta}$$

Then

$\kappa \in [0,1]$, by (3.4) we see that

$$\frac{d^*(\mathfrak{u}_n, \mathfrak{u}_{n+1})}{d^*(A,B)} \leq \frac{\varphi_{n-1} + \frac{\mathfrak{x}_2}{\mathfrak{x}_1}\psi_{n-1}}{\frac{1}{\mathfrak{x}_1} - \left(\varphi_{n-1} + \frac{\mathfrak{x}_2}{\mathfrak{x}_1}\psi_{n-1}\right)}$$

$$\leq \left(\frac{d^*(\mathfrak{u}_{n-1}, \mathfrak{u}_n)}{d^*(A,B)}\right)^{\kappa}$$

$$\leq \left(\frac{d^*(\mathfrak{u}_{n-2}, \mathfrak{u}_{n-1})}{d^*(A,B)}\right)^{\kappa^2}$$

$$\vdots$$

$$\leq \left(\frac{d^*(\mathfrak{u}_0, \mathfrak{u}_1)}{d^*(A,B)}\right)^{\kappa^n}$$

Since $\kappa \in [0,1)$, Consequently $\lim_{n\to\infty} \kappa^n = 1$ and

$$\lim_{n\to\infty} d^*(\mathfrak{u}_n, \mathfrak{u}_{n+1}) = d^*(A,B) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.5)$$

Now, let $\{\mathfrak{u}_{2n_j}\}$ be a convergent subsequence of $\{\mathfrak{u}_{2n}\}$ and $\mathfrak{u}_{2n_j} \to \mathfrak{u}$ as $j \to \infty$ for some

$\mathfrak{u} \in A$. By definition 3.1, We get

$$d^*\left(\zeta\mathfrak{u}, \mathfrak{u}_{2n_j}\right) = d^*\left(\zeta\mathfrak{u}, \Gamma\mathfrak{u}_{2n_{j-1}}\right)$$

$$\leq d^*\left(\mathfrak{u}_{2n_{j-1}}, \mathfrak{u}_{2n_j}\right)^{\left(\frac{\mathfrak{x}_1}{\mu}\varphi_{2n_{j-1}} + \frac{\mathfrak{x}_2}{\mu}\psi_{2n_{j-1}}\right)} . \, d^*(A,B)^{\frac{1}{\mu}\left(1-2\left(\mathfrak{x}_1\varphi_{2n_{j-1}} + \mathfrak{x}_2\psi_{2n_{j-1}}\right)\right)}$$

$\forall\, j \in \mathbb{N}$ since $\mathfrak{u}_{2n_j} \to \mathfrak{u}$ as $n \to \infty$, Applying $\lim\ sup$ as $j \to \infty$ to the above inequality and making use of (3.5), we obtain $d^*(\zeta\mathfrak{u}, \mathfrak{u})^{\frac{1}{\mu}} \leq d^*(A,B)^{\frac{1}{\mu}}$.

Then $d^*(\zeta\mathfrak{u}, \mathfrak{u}) \leq d^*(A,B)$

On the other hand, since

$$d^*(A,B) = \inf\{d^*(\mathfrak{u},\mathfrak{v}); \mathfrak{u} \in A, \mathfrak{v} \in B\}$$

We have $d^*(\zeta\mathfrak{u},\mathfrak{u}) \geq d^*(A,B)$, therefore $d^*(\zeta\mathfrak{u},\mathfrak{u}) = d^*(A,B)$, (i) is proved.

Conclusion (2) can be established through reasoning analogous to that used in the proof of (1). A similar line of argument holds when $\mathfrak{x}_1 \leq \mathfrak{x}_2$. Hence, the proof is complete.

3.3. Theorem

Let A and B be non-empty subsets of a $\mathfrak{b} - \mathcal{MMS}\ (\chi, d^*, \mu)$ and $\zeta: A \to B$ and $\Gamma: B \to A$ be maps. If the pair (ζ, Γ) satisfy the weak b-R-KC multiplicative rational cyclic condition then for $\mathfrak{x}_2 < \frac{1}{2}$, the sequence $\{\mathfrak{u}_n\}$ is bounded.

Proof:

Accordingly

$$\lim_{\Gamma\to\infty} d^*(\mathfrak{u}_n, \mathfrak{u}_{n+1}) = d^*(A,B)$$

Since $\{d^*(\mathfrak{u}_{2n-1}, \mathfrak{u}_{2n})\}$ is a subsequence of $\{d^*(\mathfrak{u}_n, \mathfrak{u}_{n+1})\}$, we have

$$\lim_{\Gamma\to\infty} d^*(\mathfrak{u}_{2n-1}, \mathfrak{u}_{2n}) = d^*(A,B)$$

Thus, $\{d^*(\mathfrak{u}_{2n-1}, \mathfrak{u}_{2n})\}$ is bounded. Therefore, $\exists\ M > 1 \ni$

$d^*(\mathfrak{u}_{2n-1}, \mathfrak{u}_{2n}) \leq M^{\frac{1}{\mu}}\ \ \forall\ n \in \mathbb{N}$.

Given that φ and ψ fulfill the weak b-R-KC rational cyclic condition for each $n \in \mathbb{N}$, it follows that

$$d^*(\zeta\mathfrak{u}_0, \mathfrak{u}_{2n}) = d^*(\zeta\mathfrak{u}_0, \Gamma\mathfrak{u}_{2n-1})$$

$$\leq \left(\frac{d^*(\mathfrak{u}_0,\zeta\mathfrak{u}_0).d^*(\mathfrak{u}_0,\mathfrak{u}_{2n})}{d^*(\mathfrak{u}_0,\zeta\mathfrak{u}_0).d^*(\mathfrak{u}_{2n-1},\mathfrak{u}_{2n})}\right)^{\frac{\mathfrak{x}_1}{\mu}\varphi\left(d^*(\mathfrak{u}_0,\mathfrak{u}_{2n-1})\right)} . \left(\frac{d^*(\mathfrak{u}_0,\zeta\mathfrak{u}_0).d^*(\mathfrak{u}_0,\mathfrak{u}_{2n})}{d^*(\mathfrak{u}_0,\mathfrak{u}_{2n}).d^*(\mathfrak{u}_{2n-1},\zeta\mathfrak{u}_0)}\right)^{\frac{\mathfrak{x}_2}{\mu}\psi\left(d^*(\mathfrak{u}_0,\mathfrak{u}_{2n-1})\right)}$$

$$.\, d^*(A,B)^{\left(\frac{1}{\mu}\left(1-2\left(\mathfrak{x}_1\varphi\left(d^*(\mathfrak{u}_0,\mathfrak{u}_{2n-1})\right)+\mathfrak{x}_2\psi\left(d^*(\mathfrak{u}_0,\mathfrak{u}_{2n-1})\right)\right)\right)\right)}$$

$$d^*(\zeta\mathfrak{u}_0, \mathfrak{u}_{2n}) = d^*\left(\zeta\mathfrak{u}_0, \Gamma\mathfrak{u}_{2n_1}\right)$$

$$\leq d^*\left(\mathfrak{u}_{2n_1}, \mathfrak{u}_{2n}\right)^{\left(\frac{\mathfrak{x}_1}{\mu}\varphi_{2n_1}+\frac{\mathfrak{x}_2}{\mu}\psi_{2n_1}\right)} . d^*(A,B)^{\frac{1}{\mu}\left(1-2\left(\mathfrak{x}_1\varphi_{2n_1}+\mathfrak{x}_2\psi_{2n_1}\right)\right)}$$

$$\leq M^{\frac{1}{\mu}(\mathfrak{x}_1+\mathfrak{x}_2)} . d^*(A,B)$$

Choose $\mathfrak{X} := M^{(\mathfrak{x}_1+\mathfrak{x}_2)} . d^*(A,B)$, then

$$d^*(\zeta\mathfrak{u}_0, \mathfrak{u}_{2n}) \leq \mathfrak{X}^{\frac{1}{\mu}} \quad \text{.........................} \quad (3.6)$$

Furthermore,

$$d^*(\zeta\mathfrak{u}_0, \mathfrak{u}_{2n+1}) \leq \{d^*(\zeta\mathfrak{u}_0, \mathfrak{u}_{2n}) . d^*(\mathfrak{u}_{2n}, \mathfrak{u}_{2n+1})\}^{\mu}$$

$$\leq (M.\mathfrak{X})^{\mu.\frac{1}{\mu}}$$

$$\leq M.\mathfrak{X} \quad \text{....................} \quad (3.7)$$

Hence from (3.6) and (3.7) $\forall\ n \in \mathbb{N}$, it results that $d^*(\zeta\mathfrak{u}_0, \mathfrak{u}_n) \leq M.\mathfrak{X}$ which means that $\{\mathfrak{u}_n\}$ is bounded.

4. Application

Consider $\chi = C[0,1]$ representing the set of all real-valued continuous functions defined on the interval [0,1]. Let χ be endowed with the $\mathfrak{b} - \mathcal{MMS}$ below,

$$d^*(\mathfrak{p}, \mathfrak{q}) = sup\left\{\left|\frac{\mathfrak{p}(t)}{\mathfrak{q}(t)}\right|^2 : 0 \leq t \leq 1\right\}, \mathfrak{p}, \mathfrak{q} \in \chi.$$

Then (χ, d^*, μ) is a complete $\mathfrak{b} - \mathcal{MMS}$ with parameter $\mu = 2$. Next, we turn our attention to integral equations,

$$\mathfrak{p}(t) = \int_0^1 \mathcal{F}(t, \mathfrak{s})\, \mathfrak{x}_1(t, \mathfrak{s}, \mathfrak{p}(\mathfrak{s}))^{d\mathfrak{s}}$$

$$\mathfrak{q}(t) = \int_0^1 \mathcal{F}(t, \mathfrak{s})\, \mathfrak{x}_2(t, \mathfrak{s}, \mathfrak{q}(\mathfrak{s}))^{d\mathfrak{s}}$$

Where $\mathcal{F}: [0,1] \times [0,1] \to \mathbb{R}$ and $\mathfrak{x}_1, \mathfrak{x}_2 = [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Take that:

(1) For all $t, \mathfrak{s} \in [0,1]$ and $\mathfrak{p}, \mathfrak{q} \in \chi$, this yields:

$$0 \leq \int_0^1 \mathcal{F}(t, \mathfrak{s})\, \mathfrak{x}_1(t, \mathfrak{s}, \mathfrak{p}(\mathfrak{s}))^{d\mathfrak{s}} \leq \frac{1}{3}$$

and

$$0 \leq \int_0^1 \mathcal{F}(t, \mathfrak{s})\, \mathfrak{x}_2(t, \mathfrak{s}, \mathfrak{q}(\mathfrak{s}))^{d\mathfrak{s}} \leq \frac{1}{2}$$

Define:

A = $\left\{\mathfrak{p} \in \chi \mid 0 \leq \mathfrak{p}(t) \leq \frac{1}{2}\right\}$ and

B = $\left\{\mathfrak{q} \in \chi \mid 0 \leq \mathfrak{q}(t) \leq \frac{1}{3}\right\}$,

Then $d^*(A, B) = \frac{1}{4}$

(2) For all $t, \mathfrak{s} \in [0,1]$, we have

$$\max_{0 \leq t \leq 1} \int_0^1 |\mathcal{F}(t, \mathfrak{s})|^{2^{\,d\mathfrak{s}}} \leq d^*(A, B)^{\frac{1}{2}}$$

(3) For all $t, \mathfrak{s} \in [0,1]$ and $\mathfrak{p}, \mathfrak{q} \in \chi$, thus:

$$\left|\frac{\mathfrak{x}_1(t, \mathfrak{s}, \mathfrak{p}(\mathfrak{s}))}{\mathfrak{x}_2(t, \mathfrak{s}, \mathfrak{q}(\mathfrak{s}))}\right| \leq \left|\frac{\mathfrak{p}(\mathfrak{s})}{\mathfrak{q}(\mathfrak{s})}\right|^{\frac{1}{\sqrt{2}}}$$

$$\leq \left|\frac{\mathfrak{p}(\mathfrak{s})}{\zeta \mathfrak{p}(\mathfrak{s})}\right|$$

$$\leq \left|\frac{\mathfrak{p}(\mathfrak{s})}{\Gamma\mathfrak{q}(\mathfrak{s})}\right|$$

Let $\zeta: A \to B$ and $\Gamma: B \to A$ be mappings defined by

$$\zeta\mathfrak{p}(\mathfrak{t}) = \int_0^1 \mathcal{F}(\mathfrak{t},\mathfrak{s})\, \mathfrak{x}_1\big(\mathfrak{t},\mathfrak{s},\mathfrak{p}(\mathfrak{s})\big)^{d\mathfrak{s}}$$

$$\Gamma\mathfrak{q}(\mathfrak{t}) = \int_0^1 \mathcal{F}(\mathfrak{t},\mathfrak{s})\, \mathfrak{x}_2\big(\mathfrak{t},\mathfrak{s},\mathfrak{q}(\mathfrak{s})\big)^{d\mathfrak{s}}$$

Consider $\varphi(t) = \psi(t) = \frac{\mathfrak{t}}{2}$ and fix $\mathfrak{x}_1 = \mathfrak{x}_2 = \frac{1}{4}$. We aim to establish that the pair of operators ζ and Γ meet the hypotheses of Theorem 3.2 for any of $\mathfrak{p} \in A$ and $\mathfrak{q} \in B$, $\mathfrak{p} \neq \mathfrak{s}\mathfrak{q}$, we have

$$d^*(\zeta\mathfrak{p},\Gamma\mathfrak{q}) = \sup_{0\le\mathfrak{t}\le1}\left\{\left|\frac{\zeta\mathfrak{p}(\mathfrak{t})}{\Gamma\mathfrak{q}(\mathfrak{t})}\right|^2\right\}$$

$$= \sup_{0\le\mathfrak{t}\le1}\left\{\left|\frac{\int_0^1 \mathcal{F}(\mathfrak{t},\mathfrak{s})\, \mathfrak{x}_1\big(\mathfrak{t},\mathfrak{s},\mathfrak{p}(\mathfrak{s})\big)^{d\mathfrak{s}}}{\int_0^1 \mathcal{F}(\mathfrak{t},\mathfrak{s})\, \mathfrak{x}_2\big(\mathfrak{t},\mathfrak{s},\mathfrak{q}(\mathfrak{s})\big)^{d\mathfrak{s}}}\right|^2\right\}$$

$$\leq \sup_{0\le\mathfrak{t}\le1}\left\{\int_0^1 |\mathcal{F}(\mathfrak{t},\mathfrak{s})| \left(\left|\frac{\mathfrak{x}_1\big(\mathfrak{t},\mathfrak{s},\mathfrak{p}(\mathfrak{s})\big)}{\mathfrak{x}_2\big(\mathfrak{t},\mathfrak{s},\mathfrak{q}(\mathfrak{s})\big)}\right|^{d\mathfrak{s}}\right)^2\right\}$$

$$\leq \sup_{0\le\mathfrak{t}\le1}\left\{\int_0^1 |\mathcal{F}(\mathfrak{t},\mathfrak{s})|^{2d\mathfrak{s}} \int_0^1 \left|\frac{\mathfrak{x}_1\big(\mathfrak{t},\mathfrak{s},\mathfrak{p}(\mathfrak{s})\big)}{\mathfrak{x}_2\big(\mathfrak{t},\mathfrak{s},\mathfrak{q}(\mathfrak{s})\big)}\right|^{2d\mathfrak{s}}\right\}$$

$$\leq d^*(A,B)^{\frac{1}{2}} \int_0^1 \left(\left|\frac{\mathfrak{p}(\mathfrak{s})}{\mathfrak{q}(\mathfrak{s})}\right|^2 . \left|\frac{\mathfrak{p}(\mathfrak{s})}{\zeta\mathfrak{p}(\mathfrak{s})}\right|^2 . \left|\frac{\mathfrak{p}(\mathfrak{s})}{\Gamma\mathfrak{q}(\mathfrak{s})}\right|^2\right)^{\frac{d\mathfrak{s}}{2}}$$

$$\leq d^*(A,B)^{\frac{1}{2}} \left(d^*(\mathfrak{p},\mathfrak{q})^{\frac{1}{2}} .\ d^*(\mathfrak{p},\zeta\mathfrak{p}) .\ d^*(\mathfrak{p},\Gamma\mathfrak{q})\right)$$

$$\leq d^*(A,B)^{\frac{1}{2}} \left(d^*(\mathfrak{p},\mathfrak{q})^{\frac{1}{2}} .\ d^*(\mathfrak{p},\zeta\mathfrak{p}) .\ d^*(\mathfrak{p},\Gamma\mathfrak{q})\right)\left(\frac{1}{d^*(\mathfrak{p},\zeta\mathfrak{p}).d^*(\mathfrak{q},\Gamma\mathfrak{q})} . \frac{1}{d^*(\mathfrak{p},\Gamma\mathfrak{q}).d^*(\mathfrak{q},\zeta\mathfrak{p})}\right)$$

$$\leq d^*(A,B)^{\frac{1}{2}+[\varphi(d^*(\mathfrak{p},\mathfrak{q}))+\psi(d^*(\mathfrak{p},\mathfrak{q}))]} . \left(\frac{d^*(\mathfrak{p},\zeta\mathfrak{p}) .\ d^*(\mathfrak{p},\Gamma\mathfrak{q})}{d^*(\mathfrak{p},\zeta\mathfrak{p}). d^*(\mathfrak{q},\Gamma\mathfrak{q})}\right)^{\varphi(d^*(\mathfrak{p},\mathfrak{q}))} . \left(\frac{d^*(\mathfrak{p},\zeta\mathfrak{p}) .\ d^*(\mathfrak{p},\Gamma\mathfrak{q})}{d^*(\mathfrak{p},\Gamma\mathfrak{q}). d^*(\mathfrak{q},\zeta\mathfrak{p})}\right)^{\psi(d^*(\mathfrak{p},\mathfrak{q}))}$$

$$\leq \left(\frac{1}{4}\right)^{\frac{1}{2}+[\varphi(d^*(\mathfrak{p},\mathfrak{q}))+\psi(d^*(\mathfrak{p},\mathfrak{q}))]} . \left(\frac{d^*(\mathfrak{p},\zeta\mathfrak{p}) .\ d^*(\mathfrak{p},\Gamma\mathfrak{q})}{d^*(\mathfrak{p},\zeta\mathfrak{p}). d^*(\mathfrak{q},\Gamma\mathfrak{q})}\right)^{\varphi(d^*(\mathfrak{p},\mathfrak{q}))} . \left(\frac{d^*(\mathfrak{p},\zeta\mathfrak{p}) .\ d^*(\mathfrak{p},\Gamma\mathfrak{q})}{d^*(\mathfrak{p},\Gamma\mathfrak{q}). d^*(\mathfrak{q},\zeta\mathfrak{p})}\right)^{\psi(d^*(\mathfrak{p},\mathfrak{q}))}$$

As a result, the prerequisites of Theorem 3.2 are satisfied when s = 2, $\mathfrak{x}_1 = \mathfrak{x}_2 = \frac{1}{4}$ and $\psi(\mathfrak{t}) = \varphi(\mathfrak{t}) = \frac{\mathfrak{t}}{2}$.

5. Conclusion

In this study, we have introduced novel rational cyclic conditions within the framework of b-multiplicative metric spaces, thereby extending classical results in fixed point and best proximity point theory. By establishing several best proximity point theorems for mappings that satisfy these generalized conditions. The inclusion of an application to a nonlinear integral equation demonstrates the practical utility of our results in solving real-world problems modeled by such equations. The future work can explore the extension of these rational cyclic conditions to other generalized metric-like spaces such as cone metric spaces, fuzzy metric spaces, or G-metric spaces.

Compliance with ethical standards

Disclosure of conflict of interest

The authors declare that there is no conflict of interests.

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