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(REVIEW ARTICLE)

Cosmic expansion of Bianchi type-III wet dark fluid model in $f(R)$ theory of gravity

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Abstract

In this study, our primary focus was the examination of the Bianchi type-III metric in the context of $f(R)$ gravitational theory incorporating a wet dark fluid. We have determined that the metric solution corresponds to an expanding universe, achieved by assuming a negative constant deceleration parameter and engaging a power-law relationship. Furthermore, we have conducted a visual assessment of the metric's dynamic and geometric characteristics using graphical representations.

Keywords: Banchi type-III Model; Wet dark fluid; $f(R)$ gravity; Deceleration parameter

1. Introduction

Einstein's general theory of relativity provides explanations for a wide range of gravitational phenomena, and its validity has been rigorously confirmed through laboratory experiments. The outcomes derived from general relativity align with the principles of the standard model of particle physics and have been successfully applied to phenomena at the scale of our solar system. But, certain significant challenges, such as understanding the accelerated expansion of the universe, addressing the cosmological constant, and resolving the enigma of dark energy, remain unresolved according to recent research [1, 2].

In an attempt to address these unresolved questions, various models have been proposed. One such endeavor is the $f(R)$ gravity model [3-5]. This model includes an extension of the Einstein-Hilbert action to introduce modifications to the gravitational field. Specifically, it replaces the Ricci scalar with a function of the Ricci scalar. This modification of gravity delivers a theoretical explanation for the presently observed accelerated expansion of the universe [6-7].

Santhi *et al.* [8] examined a viscous string Bianchi type-III cosmological model in the context of $f(R)$ theory of gravitation. They explored the model under two conditions: one with the presence of a cosmic string and the other one is without it. They derived solutions for this model by utilizing the Hubble parameter. Shamir [9] delved into a comprehensive examination of the precise vacuum solutions pertaining to the cosmological models of Bianchi types I, III, and Kantowski-Sachs within the framework of $f(R)$ gravity's metric formulation. This analysis was conducted under the consideration that the expansion scalar θ maintains a direct proportionality with the shear scalar σ. The resultant conclusion drawn from the research postulates that, over time, the volume of the universe exhibits a discernible, persistent expansion. Numerous researchers have undertaken endeavors to investigate Bianchi type-III and various cosmological models within the framework of the $f(R)$ theory of gravity [10-15].

The wet dark fluid is a highly favored subject of study for numerous researchers across various contexts. Samantha [16] studied the universe containing dark energy (DE) derived from a wet dark fluid (WDF) within the context of $f(R)$

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theory of gravity. Through the utilization of a novel equation of state (EoS) for the DE component, noted that the universe was progressively becoming more isotropic when influenced by the presence of this wet dark fluid. Additionally, Samantha explored concepts such as look-back time, luminosity distance, and redshift. Adhav *et al.* [17] investigated the wet dark fluid model within the context of general theory of relativity, specifically they focus on the Bianchi type-III cosmological model. Their findings led to the conclusion that the Universe does not tend to become more isotropic. Additionally, their study delved into the examination of the dynamical properties of the Universe. Nimkar and Pund [18] have conducted research on Ruban's background within the framework of the wet dark fluid model in the context of the General Theory of Relativity. Several other researchers did their work devotedly in the area of wet dark fluid in different context. [19-22].

Motivated by the above study, our current research focuses on Bianchi type-III universe with wet dark fluid in $f(R)$ theory of gravity. The work is divided into different sections and arranged as introduction, wet dark fluid, basic formation of $f(R)$ gravity, the solution of field equation with cosmological model, graphical approach, result and references.

2. Wet dark fluid

In 2005 Holman and Naidu [23] investigated a homogeneous case of Friedmann Robertson –Walker model utilizing wet dark fluid (WDF) as dark energy model. The spirit of generalized Chaplygin gas was studied by Gorini *et al.* [24]. A physically motivated equation of state (EoS) is presented with properties addressing the dark energy model. The inspiration arises from the empirical EoS given by Hayward [25].

The simplest form of equation of state of WDF is given by

$$P_{WDF} = \gamma\left(\rho_{WDF} - \rho^*\right) \quad \text{.................(1)}$$

where P_{WDF} , ρ_{WDF} represents a wet dark fluid presure and energy density respectively and $\gamma > 0$, $\rho^* > 0$ are constants. Without loss of generality we restrict our self to $0 < \gamma < 1$. It finds an inspiration in the fact that it gives as a suitable approximation for various fluid as water, where the initial molecular attraction permit for the appearance of negative pressure. Here we note that if c_s denotes the adiabatic sound speed in Wet dark fluid, then $\gamma = c_s^2$ given by Babichev [26].

To find the energy density for WDF, we use the energy conservation equation

$$\dot{\rho}_{WDF} + 3H\left(p_{WDF} + \rho_{WDF}\right) = 0 \qquad , \text{.................. (2)}$$

On using equation of state (1) and $3H = \frac{\dot{V}}{V}$ in equation (2), we get

$$\rho_{WDF} = \frac{\gamma}{1+\gamma}\rho^* + \frac{k}{V^{(1+\gamma)}} \quad \text{.................. (3)}$$

where k is an integration constant and V is a volume expansion.

Wet dark fluid includes two components: one component behaves as a cosmological constant whereas the other component represents the red shift as a standard fluid, for which the equation of state is given by $p = \gamma\rho$. It is easy to show that, if we take $k > 0$, the fluid will not break the strong energy condition given by $p + \rho \geq 0$.

$$p_{WDF} + \rho_{WDF} = (1+\gamma)\,\rho_{WDF} - \gamma\rho^*$$

$$p_{WDF} + \rho_{WDF} = (1+\gamma)\frac{k}{V^{(1+\gamma)}} \geq 0 \quad \text{................... (4)}$$

3. Basic formation of $f(R)$ gravity

The $f(R)$ theory of gravity was proposed by Buchdahl [27] in 1970 in which he used ϕ instead of f . The Generalization in General Theory of Relativity is $f(R)$ theory of Gravity. The "Metric Approach", "Palatine Approach" and "Affine $f(R)$ gravity" are the three main approaches in $f(R)$ theory of gravity.

The action for $f(R)$ gravity is given by

$$S = \frac{1}{2}\int \sqrt{-g}\left(f(R) + L_M\left(g_{ij}, \psi\right)\right)d^4x \quad , \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

where $f(R)$ is a general function of the Ricci scalar R , g is the determinant of metric g_{ij} and L_m is metric Lagrangian which is depends on the metric g_{ij} and ψ the matter field. The field equations in metric formalism for $f(R)$ gravity by varying the action given in equation (1) with respect to the metric g_{ij} is given by

$$F(R)R_{ij} - \frac{1}{2}f(R)g_{ij} + [g_{ij}g^{ij}\nabla_i\nabla_j - \nabla_i\nabla_j]F(R) = T_{ij} , \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

where $F = \frac{df}{dR}$, ∇ is an covariant derivative, $g^{ij}\nabla_i\nabla_j$ is an D'Alemberts operator and T_{ij} is standard energy momentum tensor obtained by using Lagrangian L_m.

4. Metric and dynamical parameters

The LRS Bianchi type-I cosmological model given by

$$ds^2 = dt^2 - A_1^2(t)\,dx^2 - A_2^2(t)e^{2nx}dy^2 - A_3^2(t)dz^2 \ , \ldots\ldots\ldots (7)$$

where A_1^2, A_2^2, A_3^2 are the function of cosmic time t only and $n > 0$ be the constant.

For the wet dark fluid, the energy momentum tensor is, Adhav *et al.* [9]

$$T_{ij} = \left(p_{WDF} + \rho_{WDF}\right)u_i u_j - p_{WDF}\, g_{ij} \ , \ldots\ldots\ldots (8)$$

with $x^i = (1, 0, 0, 0)$ $u^i = (0, 0, 0, 1)$ being the four velocities, for which $u^i u_i = 1, u^i \nabla_j u_i = 0$.

$$_{WDF}T_1^1 = {}_{WDF}T_2^2 = {}_{WDF}T_3^3 = -p_{WDF} , \qquad {}_{WDF}T_4^4 = \rho_{WDF} \quad \ldots\ldots (9)$$

The dynamical parameters related to cosmological model (7) are defined as follows:

The special volume V in the form of average scale factor $a(t)$ of model is defined as

$$V = a(t)^3 = A_1 A_2 A_3 \ \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

The mean Hubble parameter H is defined as

$$H = \frac{\dot{a}}{a} = \frac{1}{3}\left(H_x + H_y + H_z\right) = \frac{1}{3}\left(2\frac{\dot{A}_1}{A_1} + \frac{\dot{A}_3}{A_3}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

where H_x, H_y, H_z are directional Hubble parameter in the direction of x, y, z respectively.

To discuss the anisotropy of universe, the anisotropic parameter Δ is define as

$$\Delta = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2 \quad \text{.................} \ (12)$$

The expansion parameter θ is defined as

$$\theta = 3H \quad \text{.................} \ (13)$$

The deceleration parameter $q(t)$ is defined as

$$q = \frac{d}{dt}\left(\frac{1}{H}\right) - 1, \text{.................} \ (14)$$

It grants the acceleration of the universe for $-1 \le q(t) < 0$, deceleration of the universe for the $q(t) > 0$ and constant rate of expansion for $q(t) = 0$.

The Ricci scalar R of model is given by

$$R = 2\left[2\frac{\ddot{A}_1}{A_1} + \frac{\ddot{A}_3}{A_3} + \left(\frac{\dot{A}_1}{A_1}\right)^2 + 2\frac{\dot{A}_1\dot{A}_3}{A_1A_3} - \frac{n}{A_1^2}\right] \text{.................} \ (15)$$

In the presence of wet dark fluid as source given in equation (8), the field equation (6) analogous with metric given in equation (7) gives the set of linearly independent equations as

$$\frac{\ddot{A}_1}{A_1} + \frac{\dot{A}_1\dot{A}_2}{A_1A_2} + \frac{\dot{A}_1\dot{A}_3}{A_1A_3} - \frac{n}{A_1^2} - \frac{f(R)}{2F} + \left(\frac{\dot{A}_2}{A_2} + \frac{\dot{A}_3}{A_3}\right)\frac{\dot{F}}{F} + \frac{\ddot{F}}{F} + \frac{p_{WDF}}{F} = 0 \quad \text{.................} \ (16)$$

$$\frac{\ddot{A}_2}{A_2} + \frac{\dot{A}_1\dot{A}_2}{A_1A_2} + \frac{\dot{A}_1\dot{A}_3}{A_1A_3} - \frac{n}{A_1^2} - \frac{f(R)}{2F} + \left(\frac{\dot{A}_1}{A_1} + \frac{\dot{A}_3}{A_3}\right)\frac{\dot{F}}{F} + \frac{\ddot{F}}{F} + \frac{p_{WDF}}{F} = 0, \quad \text{.................} \ (17)$$

$$\frac{\ddot{A}_3}{A_3} + \frac{\dot{A}_1\dot{A}_3}{A_1A_3} + \frac{\dot{A}_2\dot{A}_3}{A_2A_3} - \frac{f(R)}{2F} + \left(\frac{\dot{A}_2}{A_2} + \frac{\dot{A}_1}{A_1}\right)\frac{\dot{F}}{F} + \frac{\ddot{F}}{F} + \frac{p_{WDF}}{F} = 0 \quad \text{.................} \ (18)$$

$$\frac{\ddot{A}_1}{A_1} + \frac{\ddot{A}_2}{A_2} + \frac{\ddot{A}_3}{A_3} - \frac{f(R)}{2F} + \left(\frac{\dot{A}_3}{A_3} + \frac{\dot{A}_2}{A_2} + \frac{\dot{A}_1}{A_1}\right)\frac{\dot{F}}{F} - \frac{\rho_{WDF}}{F} = 0, \quad \text{.................} \ (19)$$

$$\left(\frac{\dot{A}_1}{A_1} - \frac{\dot{A}_2}{A_2}\right)F = 0, \quad \text{.................} \ (20)$$

here the overhead dots represent a differentiation with respect to the cosmic time t.

5. Solution of field equation

On taking the integration of equation (20), without loss of generality with the choice of integration constant equal to one, we get

$$A_1 = A_2 \text{ , ... (21)}$$

Using equation (21) with equations (16-20), we get

$$\frac{\ddot{A}_1}{A_1}+\frac{\dot{A}_1^2}{A_1^2}+\frac{\dot{A}_1\dot{A}_3}{A_1A_3}-\frac{n}{A_1^2}-\frac{1}{2F}f(R)+\left(\frac{\dot{A}_1}{A_1}+\frac{\dot{A}_3}{A_3}\right)\frac{\dot{F}}{F}+\frac{\ddot{F}}{F}+\frac{p_{WDF}}{F}=0 \text{ .... (22)}$$

$$\frac{\ddot{A}_3}{A_3}+2\frac{\dot{A}_1\dot{A}_3}{A_1A_3}-\frac{1}{2F}f(R)+2\frac{\dot{A}_1}{A_1}\frac{\dot{F}}{F}+\frac{\ddot{F}}{F}+\frac{p_{WDF}}{F}=0 \text{ .... (23)}$$

$$2\frac{\ddot{A}_1}{A_1}+\frac{\ddot{A}_3}{A_3}-\frac{1}{2F}f(R)+\left(\frac{\dot{A}_3}{A_3}+2\frac{\dot{A}_1}{A_1}\right)\frac{\dot{F}}{F}-\frac{\rho_{WDF}}{F}=0 \text{ .... (24)}$$

here, the set of equations (22) –(24) forms a system of highly non-linear independent equations with six unknowns $A_1, A_2, F, f(R), p_{WDE}, \rho_{WDF}$. Hence to solve the above system of non-linear independent equations we consider the special law of variation of Hubble parameter given by Berman [28] which leads to the constant deceleration parameter of model.

$$q=-\frac{\ddot{a}a}{\dot{a}^2}=\text{constant, ... (24)}$$

where a is average scale factor depending on the cosmic time *t*.

Here the deceleration parameter is taken negative for purpose of accelerating model of the universe.

On solving equation (24), it gives

$$a=(m_1 t+m_2)^{\frac{1}{1+q}} \text{ ,.................. (25)}$$

where $m_1 \neq 0$ and m_2 are integral constants. $(1+q)>0$ is the condition of expansion given by equation (24).

The relation between shear scalar is proportional to the scale expansion, which gives the relation between the metric potentials, given by [29]

$$A_1 = A_3^m \text{ ,........(26)}$$

where, $m \neq 0$ is constant which preserve the anisptropy of metric.

The power Law relation between F and a presented by Sharif and Shamir [30] is given by

$$F=\varsigma a^k \text{ ,.................. (27)}$$

where k is arbitrary constant and ς is proportionality constant.

Without loss of generality we take $\varsigma=1$. Therefore equation (27) reduces to

$$F=a^k \quad \text{, ... (28)}$$

where k is constant.

Using relation (11) and (28), we get

$$F=\left(A_3A_1^2\right)^{\frac{k}{3}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (29)$$

Using equation (21), (25) and (26) the values of $A_1(t)$ and $A_2(t)$ is given by

$$A_1(t)=A_2(t)=(m_1t+m_2)^{\frac{3m}{(2m+1)(1+q)}} \ldots\ldots\ldots\ldots\ldots\ldots (30)$$

$$A_3(t)=(m_1t+m_2)^{\frac{3}{(2m+1)(1+q)}} \ldots\ldots\ldots\ldots\ldots\ldots (31)$$

From above equations it is seen that product of power and exponential exist in $A_1(t)$ and $A_2(t)$ respectively and it increases infinitely exponentially with cosmic time t .

Using equation (30) and (31) the metric in equation (7) can be re-written as

$$ds^2=dt^2-(m_1t+m_2)^{\frac{6m}{(2m+1)(1+q)}}\left(dx^2-e^{2nx}dy^2\right)-(m_1t+m_2)^{\frac{6}{(2m+1)(1+q)}}dz^2, \ldots\ldots\ldots\ldots\ldots\ldots (32)$$

6. Properties of cosmological model

Using equation (30) and (31) the Ricci scalar R of model is given by

$$R=\frac{-m_3}{(1+q)^2(m_1t+m_2)^2}-\frac{2n}{(m_1t+m_2)^{\frac{3m}{1+2m+q+2mq}}}, \ldots\ldots\ldots\ldots\ldots\ldots (33)$$

Using equation (29) the value of function $f(R)$ is given by

$$f(R)=\frac{2m_3(m_1t+m_2)^{\frac{-2(1+q)+k}{1+q}}}{(1+q)(2-k+2q)}-\frac{6mn(m_1t+m_2)^{\frac{k-3m+2km}{1+2m+q+2mq}}}{k-3m+2km}, \ldots\ldots\ldots\ldots\ldots\ldots (34)$$

above equation (34) shows that the nature of $f(R)$ is equivalent to the function found by Sharif *et al.* [9].

The special volume of model by using equation (10) is given by

$$V=(m_1t+m_2)^{\frac{3}{1+q}} \ldots\ldots\ldots\ldots\ldots\ldots (35)$$

On using equation (11) the mean Hubble parameter H of model is given by

$$H=\frac{m_1}{(1+q)(m_1t+m_2)}, \ldots\ldots\ldots\ldots\ldots\ldots (36)$$

On using equation (12) the anisotropic parameter Δ of model is

$$\Delta=\frac{(1+2m^2)}{(1+2m)^2}-1, \ldots\ldots\ldots\ldots\ldots\ldots (37)$$

Using equation (13) the expansion parameter θ is given by

$$\theta = \frac{3m_1}{(1+q)(m_1 t + m_2)} \quad \text{.................} (38)$$

here in the throughout discussion the value of q is taken as negative constant term.

Using equation (24) & (30)-(31), (34) the energy density of obtained model is

$$\rho_{WDF} = \frac{m_4 (m_1 t + m_2)^{\frac{-2(1+q)+k}{1+q}} - \dfrac{3mn(m_1 t + m_2)^{\frac{k-3m+2km}{1+2m+q+2mq}}}{k-3m+2km}}{m_5 + \dfrac{4km_3 m_1 (m_1 t + m_2)^{\frac{-3(1+q)+2k}{1+q}}}{(1+q)^2 (k-2(1+q))} + \dfrac{12km_1 mn (m_1 t + m_2)^{-\frac{(1+q)+2k}{(1+q)} - \frac{3m}{1+2m+q+2mq}}}{(k-3m+2km)(1+q)}}, \quad \text{.................} (39)$$

Using equation (24) and (37), the pressure of obtained model is

$$p_{WDF} = \gamma \left(\frac{m_4 (m_1 t + m_2)^{\frac{-2(1+q)+k}{1+q}} - \dfrac{3mn(m_1 t + m_2)^{\frac{k-3m+2km}{1+2m+q+2mq}}}{k-3m+2km}}{m_5 + \dfrac{4km_3 m_1 (m_1 t + m_2)^{\frac{-3(1+q)+2k}{1+q}}}{(1+q)^2 (k-2(1+q))} + \dfrac{12km_1 mn (m_1 t + m_2)^{-\frac{(1+q)+2k}{(1+q)} - \frac{3m}{1+2m+q+2mq}}}{(k-3m+2km)(1+q)}} - \rho^* \right) \quad \text{.................} (40)$$

The energy condition for wet dark fluid satisfied as

$$\rho_{WDF} + p_{WDF} = \frac{m_4 (m_1 t + m_2)^{\frac{-2(1+q)+k}{1+q}} - \dfrac{3mn(m_1 t + m_2)^{\frac{k-3m+2km}{1+2m+q+2mq}}}{k-3m+2km}}{m_5 + \dfrac{4km_3 m_1 (m_1 t + m_2)^{\frac{-3(1+q)+2k}{1+q}}}{(1+q)^2 (k-2(1+q))} + \dfrac{12km_1 mn (m_1 t + m_2)^{-\frac{(1+q)+2k}{(1+q)} - \frac{3m}{1+2m+q+2mq}}}{(k-3m+2km)(1+q)}} (1+\gamma) - \gamma\rho^* \geq , \text{.........} (41)$$

Here throughout the study we use,

$$m_3 = \frac{6(-2+q+m(-2-5m+4(1+m)q))m_1^2}{(1+2m)^2}$$

$$m_4 = 3(2k(-4+m)m + k^2(1+2m) - k(3+4m(2+m))q + 6m(2+m)(1+q)m_1^2$$
$$m_5 = \left((1+2m)^2 (1+q)^2 (k-2(1+q))\right)$$

7. Graphical representation of dynematical parameter

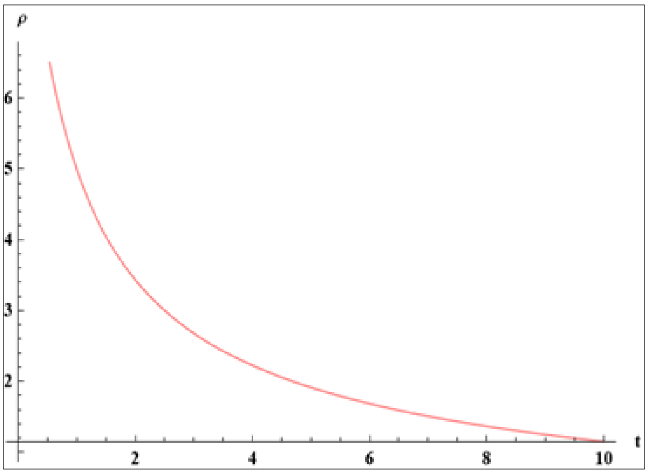

Figure 1 Energy density vs Time plotted by assuming $m_1 = 0.7, m_2 = 0.5.$ $m_2 = 0.5, m = 0.2, k = 0.1, q = -0.7, \gamma = 0.9$

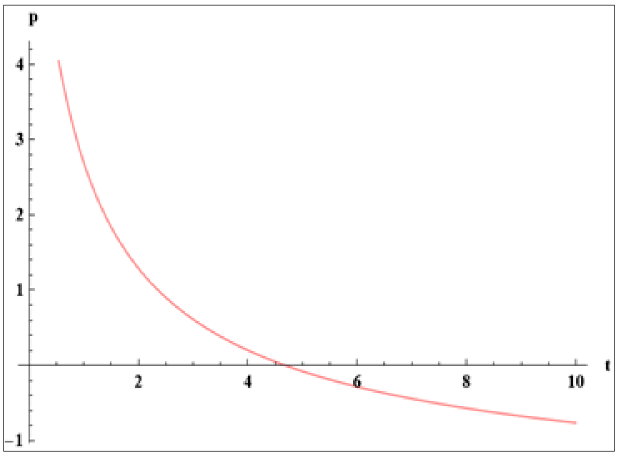

Figure 2 Pressure vs Time plotted by assuming $n = 5, m_1 = 0.7,$ $\rho^* = 2,$ $m_2 = 0.5, m = 0.2,$ $q = -0.7, \gamma = 0.9,$ $k = 0.1.$

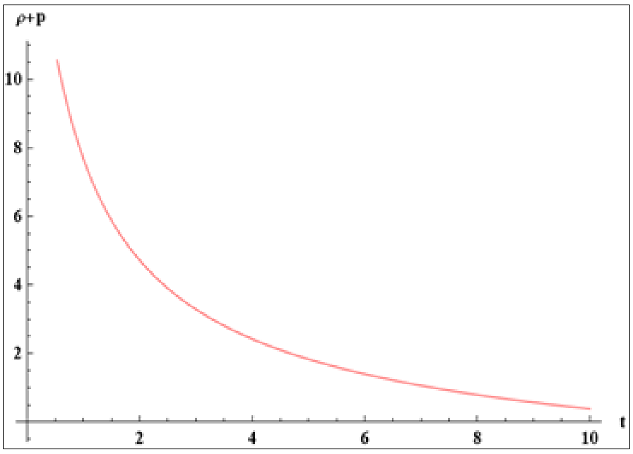

Figure 3 Sum of density and pressure vs Time plotted by assuming $m_2 = 0.5, m = 0.2, k = 0.1, q = -0.7, \gamma = 0.9, m_1 = 0.7,$ $n = 5,$ $\rho^* = 2,$

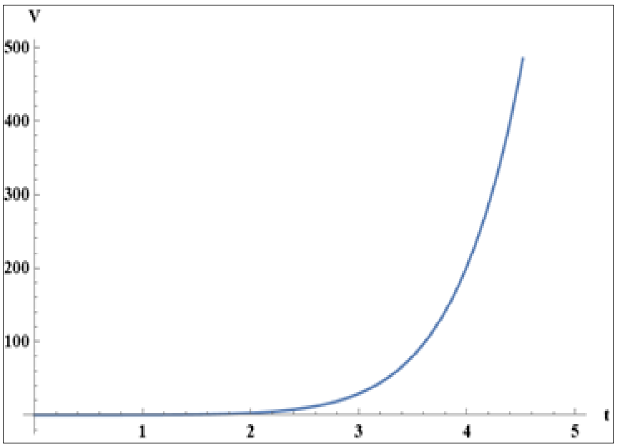

Figure 4 Volume vs Time plotted by assuming $m_1 = 0.7, m_2 = 0.5,$ $q = -0.7.$

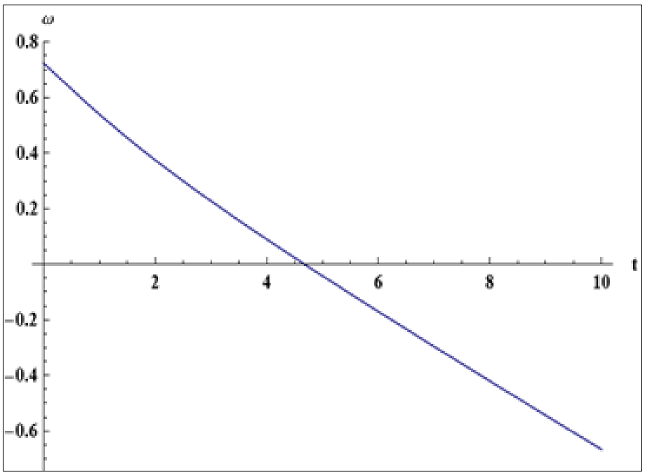

Figure 5 EoS ω vs Time plotted by assuming $n = 5, m_1 = 0.7,$ $\rho^* = 2,$ $m_2 = 0.5, m = 0.2, k = 0.1, q = -0.7, \gamma = 0.9$

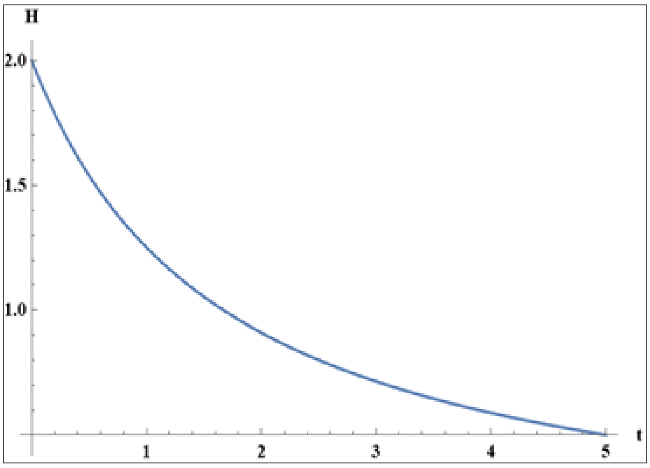

Figure 6 Hubble parameter vs Time plotted by assuming $q = -0.7,$ $m_1 = 0.7, m_2 = 0.5.$

8. Conclusion

In nutshell, we have examined a cosmic model with a constant deceleration parameter in the framework of modified gravity, focusing on a homogeneous Bianchi type-III universe filled with a wet dark fluid. Throughout the study, it is hypothesized that the deceleration parameter remains consistently negative, which is in line with the accelerating expansion of the universe. It can be observed that the nature of function of Ricci scalar R is in accordance to Sharif *et al.* [9]. The graphical observation gives, the Universe's energy density ρ is diminishing as a function of cosmic time t, gradually approaching zero and also that with cosmic time t, the pressure reduces gradually and becomes highly negative as $t \to \infty$. The volume of the universe is under exponential increment and hence the universe has observational cosmic expansion. From the figure -3, it is observed that the strong energy condition is satisfied by studied model. The nature of Hubble parameter is consistent with recent observational data.

The ratio of adiabatic sound speed c_s^2 is positive for given value of γ and hence the studied model is stable which is analogus to Katore *et al.* [31]. Figure 5 illustrates the graphical representation of the pressure-to-energy density ratio, showing a transition from positive values towards negative values, ultimately approaching -1. This observation supports the idea that the developed model will align with the flat $\Lambda-$ Cold Dark Matter (CDM) model in the future. Hence, the Universe shows the accelerating expansion. The observed result shows alignment with the latest cosmic discoveries. These suggest that our findings hold a significant value for the researchers involved in cosmic analysis of the universe in the context of $f(R)$ gravity.

Compliance with ethical standards

Disclosure of conflict of interest

No conflict of interest to be disclosed.

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