International Journal of Science and Research Archive
eISSN: 2582-8185
Cross Ref DOI: 10.30574/ijsra
Journal homepage: https://ijsra.net/

(Review Article)

An overview on graph products

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International Journal of Science and Research Archive, 2023, 10(01), 966–971

Publication history: Received on 06 September 2023; revised on 22 October 2023; accepted on 25 October 2023

Article DOI: https://doi.org/10.30574/ijsra.2023.10.1.0848

Abstract

Graph product is a very basic idea in graph theory. Graph products include a wide range of operations that join two or more existing graphs to produce new graphs with distinctive properties and uses. In this paper, we explore four forms of graph products, their characteristics, and their significance within the broader framework of graph theory.

Keywords: Cartesian product; Direct product; Strong Product; Lexicographic Product

1. Introduction

Graph theory is concerned with the mathematical structures known as graphs, which are used to represent pairwise relationships between objects. A graph is made up of vertices, which are also known as nodes or points, and edges, which are also known as links or lines. The area of mathematics known as "graph theory" studies networks of points connected by lines. Although number games and other amusing math problems gave rise to graph theory, it has developed into a sizable field of mathematical study with applications in sociology, biology, chemistry, computer technology, etc.

It is possible to pinpoint the beginning of graph theory to 1735, when Swiss mathematician Leonhard Euler found an answer to the Konigsberg bridge puzzle. An historical conundrum involving the potential of discovering a passage via each of seven bridges that cross a forked river was known as the Königsberg bridge problem. bypassing a bridge twice when passing by an island. There is no such path, according to Euler. His proof involved only references to the physical layout of the bridges, but essentially, he proved the first theorem in graph theory. The father of graph theory, as he is today referred to. The forerunners in this field were Euler and Ore.

Since then, this topic has expanded significantly. Now, there is a very strong connection between the fields of linear algebra, combinatorics, and commutative algebra and graph theory. Numerous disciplines, including sociology, electrical engineering, programming, ecology, chemical networks, and even medicine, use graphs. This field, which began innocuously enough with games and puzzles, is now a powerful tool for researching brain networks. Therefore, it is intriguing to learn more about this intriguing topic. One of the most significant applications of graph theory is Google Maps. LinkedIn and Facebook are two more examples. In order to improve their recommendation systems, Netflix and other OTT services use graph theory.

In general, mathematics looks at how a group of items behaves, how new objects are made from old ones, and so forth. New graphs are created via graph products from current ones. A key idea in graph theory is the notion of graph products. Four common graph products and some of their features are the main subjects of this study. A binary operation on a

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graph is called a graph product [1]. In more detail, it is an operation that takes two graphs, G_1 and G_2, and creates a graph H having the characteristics listed below.

“The vertex set of H is the cartesian product V (G_1)×V (G_2) where V (G_1) and V (G_2) are the vertex sets of G_1 and G_2 respectively. Two vertices (a_1, a_2) and (b_1, b_2) of H are connected by an edge, if and only if some conditions about a_1, b_1 in G_1 and a_2, b_2 in G_2 are fulfilled. The graph products differ in what exactly these conditions are. It is always about whether or not the vertices a_n, b_n in G_n are equal or connected by an edge [2].”

In this article, we also discuss several well-known parameters of product graphs. A parameter of a graph G is a numerical value that can be associated with any graph such that whenever two graphs are isomorphic, they have the same associated parameter value. ie. A graph parameter is a function ϕ : G→R where G is the set of finite graphs and R is the set of real numbers.

Graph products have broad applications in many fields. To analyse a structure, one must first generate its configuration. There are various methods available for this. One example of such a tool is the use of graph products. The creation of many kinds of structural models expands the application of product graphs [3],[4],[5]. Graph products have numerous applications in the field of computer science [6],[7],[8]. The next use for this product is to make an analytical estimation of the spectral properties of large-scale networks by approximating their structure . The creation of finite element models can also benefit from the use of weighted graph products. The literature on graph products is vast and deep, [9],[10],[11], to list a few.

The four products which we intend to investigate are Cartesian product, Direct product, Strong product and Lexicographic product. We have included definitions of graph products in respective chapters with examples. Also, some theorems and results on graph products are also discussed. We restrict ourselves to undirected graphs throughout. For all the basic definition and terminologies, we refer to [12].

2. The Cartesian product

The Cartesian product of two graphs G and H is a graph, denoted as G□H, whose vertex set is the cartesian product of V (G) and V (H), V (G)×V (H). Two vertices (g, h) and (g′, h′) are adjacent precisely if g = g′ and hh′ ∈ E(H), or gg′ ∈ E(G) and h = h′. Thus, $V\,(G\,\square H) = \{(g,h) \colon g \in V\,(G), h \in V\,(H)\}$. The edge set is given by the relation, $E(GH) = \big(E(G) \times V\,(H)\big) \cup \big(V\,(G) \times E(H)\big)$. The graphs G and H are called factors of the product $G\square H$. The Cartesian products of two paths on m and n vertices are known as grid graphs. Here, $G \times V\,(H) = \cup_{h\in H}\,(G \times \{h\})$, and every G×{h} is a copy of G. We denote it by G_h and call it a G-fiber. Analogously, V (G)×H is the union of the H-fibres = g×H.

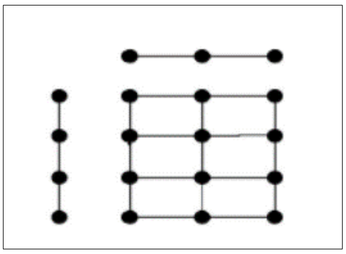

Figure 1 $P_4\square P_3$

Theorem 1. The order of Cartesian Product of graph denoted by $|G\square H|$ is the number of vertices in it.

$$|G\square H| = |G| \cdot |H|.$$

The size of G2H, denoted by ||G2H|| is the number of edges in it .

$$||G\square H|| = |H| \times ||G|| + |G| \times ||H||$$

Theorem 2. For two graphs G and H, let a∈V (G) and x∈V (H).Then

$$deg\,(a,x) = deg(a) + deg(x)$$

Theorem 3. [1] For any graphs G and H, $\chi(G\Box H) = max\{\chi(G), \chi(H)\}$.

Theorem 4. [1] If (g, h) and (g′, h′) are vertices of the Cartesian product $G\Box H$, then

$$d((g,h),(g',h')) = d(g,g') + d(h,h')$$

Theorem 5. [2] The Cartesian product $G\Box H$ is connected if and only if both factors are connected.

3. The Direct Product

The direct product of G and H is the graph, denoted as G×H, whose vertex is V (G)×V (H), and for which vertices (g, h) and(g′, h′) are adjacent precisely if gg′∈E(G) and hh′∈E(H). Thus,

$$V(G\times H) = \{(g, h) : g\in V(G) \text{ and } h\in V(H)\},$$

$$E(G\times H) = \{(g, h)(g', h') : gg'\in E(G) \text{ and } hh'\in E(H)\}.$$

Other names for the direct product that have appeared in the literature are tensor product, Kronecker product, cardinal product, relational product, cross product, conjunction, weak direct product, or categorical product. A product G×H has a loop at (g, h)if and only if both G and H have loops at g and h, respectively. Moreover, if G has no loop at g, then the H-layer $H^{(g,h)}$ is to-tally disconnected; whereas if G has a loop at g, then $H^{(g,h)}$ is isomorphic to H. Suppose (g, h) and (g′, h′) are vertices of a direct product G×H and n is an integer for which G has a g, g′- walk of length n and H has an h, h′- walk of length n. Then G×H has a walk of length n from (g, h) to (g′, h′). The smallest such n (if it exists) equals

d((g, h), (g′, h′)). If no such n exists, then d((g, h), (g′, h′)) = ∞.

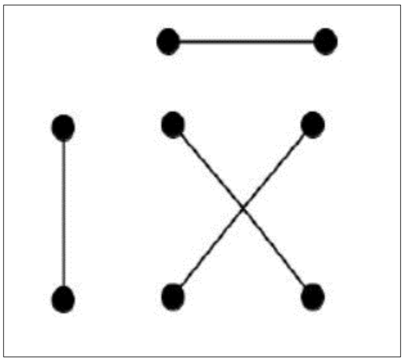

Figure 2 $P_2\times P_2$

Theorem 6: The order of direct product of graph denoted by |G×H| is the number of vertices of G×H.

$$|G\times H| = |G| \cdot |H|$$

Theorem 7. [1] [Weichsel's Theorem] Let G and H are connected nontrivial graphs. If at least one of

G or H has an odd cycle, then G×H is connected. If both G and H are bipartite, then

G×H has exactly two components.

Theorem 8. [1] For any graphs G and H, let a∈ V (G) and x ∈ V (H) then in G×H,

$$deg(a, x) = deg(a)\cdot deg(x)$$

4. The Strong Product

The strong product of G and H is the graph denoted as G⊠H and defined by

$$V(G \boxtimes H) = \{(g,h) : g \in V(G) \text{ and } h \in V(H))\},$$

$$E(G \boxtimes H) = E(G \square H) \cup E(G \times H).$$

Occasionally one also encounters the names strong direct product or symmetric composition for the strong product. Note that $G \square H$ and $G \times H$ are sub graphs of $G \boxtimes H$. The order of strong product of graph, denoted by $|G \boxtimes H|$, is equal to $|G| \cdot |H|$.

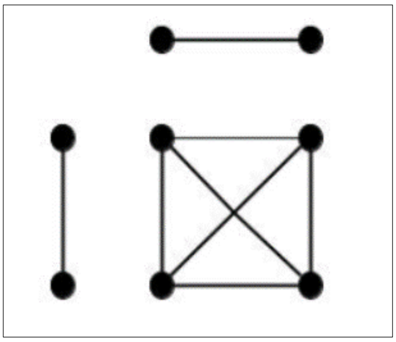

Figure 3 $P_2 \boxtimes P_2$

Theorem 9. [1] If (g, h) and (g′, h′) are vertices of a strong product $G \boxtimes H$, then

$$d((g,h),(g',h')) = \max\{d(g,g'), d(h,h')\}.$$

Theorem 10. [1] For graphs G and H, let a∈V (G) and x∈V (H), then

$$deg\,(a,x) = (deg\,(a) + 1) + (deg(x) + 1) - 1$$

Theorem 11.[1] Let G and H be graphs and Q a clique of $G \boxtimes H$. Then $Q = p_G(Q) \boxtimes p_H(Q)$, where

$p_G(Q)$ and $p_H(Q)$ are cliques of G and H, respectively.

5. The Lexicographic Product

The lexicographic product of graphs G and H is the graph denoted as $G \circ H$ whose vertex set is

V (G)×V (H) and for which (g, h)(g′, h′) is an edge of $G \circ H$ precisely if gg′ ∈ E(G) or g = g′ and hh′ ∈ E(H). This product was introduced by Harary. The lexicographic product is also known as graph substitution, a name that bears witness to the fact that $G \circ H$ can be obtained from G by substituting a copy Hg of H for every vertex g of G and then joining all vertices of H_g with all vertices of H_g if gg′ ∈ E(G). The lexicographic product is not commutative. The order of lexicographic product of graphs G and H denoted by $|G \circ H|$ is equal to $|G| \times |H|$.

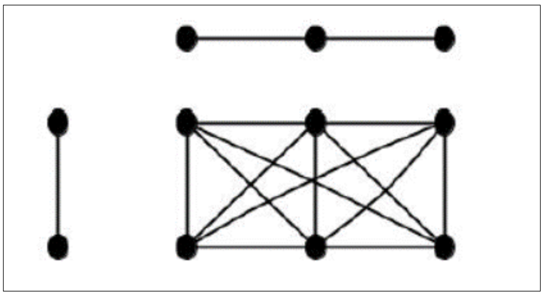

Figure 4 $P_2 \circ P_3$

Theorem 12. [1] Suppose (g, h) and (g′, h′) are two vertices of $G \circ H$. Then

$$d_{GH}((g,h),(g',h')) = \begin{cases} d_G(g,g')\ if\ g \neq g' \\ d_H(h,h')\ if\ g = g'\ and\ d_G(g) = 0 \\ min\ d_H(hh'), 2\}\ if\ g = g'\ and\ d_G(g) \neq 0 \end{cases}$$

6. Applications

Understanding the links between graphs requires a thorough understanding of graph products, which offer a way to build new graphs from older ones. These procedures are crucial to graph theory because they make it easier to explore intricate networks and systems. Additionally, research on graph products provides understanding of the combinatorial and algebraic parts of graph theory, advancing a variety of academic and practical disciplines. Graph products are important in graph theory and have uses in many different disciplines: In order to build intricate network topologies that contain particular traits and relationships, graph products are employed in network design. Graph products are used in bioinformatics to simulate and examine interactions between biological molecules, including proteins and genes. Applications of graph products in coding theory include the development of error-correcting codes and effective data transmission methods. Data visualisation employs graph products to mix multiple datasets to reveal intricate relationships.

7. Conclusion

A key idea in graph theory, graph products provide a flexible toolbox for mixing and analyzing graphs. Mathematicians, scientists, and engineers can use these operations to solve challenging issues, construct effective networks, and gain important insights into a variety of fields by knowing the numerous types of graph products, their features, and applications. The importance of graph products in graph theory rests in their capacity to establish and investigate interwoven relationships, making them the foundation of this intriguing mathematical field.

Compliance with ethical standards

Acknowledgement

The authors thank everyone who should be thanked.

Disclosure of conflict of interest

The author declares that there is no conflict of interest regarding the publication of this paper.

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