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On certain properties of analytic functions

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Abstract

In this paper, we extend the work done by Zhi-Gang et al and Owa et al for certain class of analytic functions to another subclass of analytic function. We obtain some properties related to convexity, extreme points and radius of univalency.

Keywords: Univalency; Convex; Extreme points; Radius of univalency

1. Introduction

1.1. Definition and Preliminaries

Let A denote the class of functions

$$f(z) = z + \sum_{\kappa=2}^{\infty} a_{\kappa} z^{\kappa} \qquad \text{..................1.1}$$

which is analytic in the unit disc $U = \{z \in C : |z| < 1\}$. Let P be the class of functions

$$p(z) = 1 + \sum_{\kappa=1}^{\infty} p_{\kappa} z^{\kappa} \qquad \text{.......................1.2}$$

which is also analytic in the unit disc U and have positive real part. We let $A(\beta)$ be

$f(z) \in A$ which was defined in [1] as

$$A(\beta) = \left\{ f(z) \in A : f(z)^{\beta} = z^{\beta} + \sum_{\kappa=2}^{\infty} \beta\, a_{\kappa} z^{\beta+\kappa-1} \right\} \qquad \text{..............1.3}$$

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The extreme points, coefficient bounds and radius of univalency of $Q(\alpha,\tau,\gamma)$, a subclass of A were studied in [2] where

$$Q\left(\alpha,\tau,\gamma\right)=\left\{f(z)\in A:\Re\left\{\alpha\frac{f(z)}{z}+\tau f(z)'\right\}>\gamma,z\in U\right\}$$ ……………1.4

and $\alpha,\tau,\gamma>0$, $0\le\gamma<\alpha+\tau\le1$. The authors [2] obtained Theorem 1.1 as follows:

1.2. Theorem 1.1

[2]A function $f(z)\in Q(\alpha,\tau,\gamma)$ *if and only if* $f(z)$ *can be expressed as*

$$f(z)=\frac{1}{\alpha+\tau}\int_{|x|}\left[(2\gamma-\alpha-\tau)z+2(\alpha+\tau-\gamma)\sum_{\kappa=0}^{\infty}\frac{(\alpha+\tau)x^{\kappa}z^{\kappa+1}}{(\kappa+1)\tau+\alpha}\right]d\mu(x),$$ ……………1.5

where $\mu(x)$ is the probability measure defined on $X=\{x:|x|=1\}$. For fixed α,τ *and* γ; $Q(\alpha,\tau,\gamma)$ and the probability measure μ defined on X are one-to-one by expression. (1.5)

In 2007, Owa et.al[8] studied the properties of another class of function applying higher derivatives and they obtained some interesting results. Previously, Saitoh [3] and Owa [4, 5] had worked on the relative properties of the class $Q(1-\tau,\tau,\gamma)$.

In the present paper we extend the work done by [2] on $Q(\alpha,\tau,\gamma)$ to $Q^{\beta}(\alpha,\tau,\gamma)$ where

$$Q^{\beta}(\alpha,\tau,\gamma)=\left\{f(z)^{\beta}\in A(\beta):\mathcal{R}\left\{\frac{\tau\left(f(z)^{\beta}\right)'}{\left(z^{\beta}\right)'}+\frac{\alpha f(z)^{\beta}}{z^{\beta}}\right\}>\gamma,z\in U\right\}$$ ………………..1.6

and $\alpha,\tau,\beta>0$, also $0\le\gamma<\alpha+\tau\le1$.

We will also show that the class $Q^{\beta}(\alpha,\tau,\gamma)$ is convex, using the method of Owa et.al [8].

1.3. Lemma 1.2

[6] $p\in P$ *if and only if there is probability measure* μ *on* X *such that*

$$p(z)=\int_{|x|=1}\frac{1+xz}{1-xz}d\mu(x),(|z|<1)\ \ and\ \mathcal{X}=\{x:|x|=1\}$$ ……………..*1.7*

The correspondence between P and the set of probability measure μ *on* X *given by [7] is one-to-one.*

1.4. Remark 1.1

$Q^{\beta}(\alpha,\tau,\gamma)$ *is convex.*

Proof. Let $f(z)^{\beta}\in Q^{\beta}(\alpha,\tau,\gamma)$ and $g(z)\in Q^{\beta}(\alpha,\tau,\gamma)$, we define

$$\mathfrak{F}(z)=(1-t)f(z)^{\beta}+tg(z)^{\beta},\ 0\le t\le1$$ ………………..(1.8)

Then,

$$\Re\left\{\frac{\tau\mathfrak{F}'(z)}{\left(z^{\beta}\right)'}+\frac{\alpha\mathfrak{F}(z)}{z^{\beta}}\right\}$$

$$= \Re\left\{ \frac{(1-t)\left(f(z)^{\beta}\right)' + t\left(g(z)^{\beta}\right)'}{\left(z^{\beta}\right)'} + \frac{\alpha(1-t)f(z)^{\beta} + tg(z)^{\beta}}{z^{\beta}} \right\}$$

$$- (1-t)\,\Re\left\{ \frac{\tau\left(f(z)^{\beta}\right)'}{\left(z^{\beta}\right)'} + \frac{\alpha f(z)^{\beta}}{z^{\beta}} \right\} + t\Re\left\{ \frac{\tau\left(g(z)^{\beta}\right)'}{\left(z^{\beta}\right)'} + \frac{g(z)^{\beta}}{z^{\beta}} \right\}$$

$$> \ (1-t)\ \gamma \ + \ t\gamma \ = \ \gamma.$$

Therefore $\mathfrak{F} \in Q^{\beta}(\alpha, \tau, \gamma)$.

2. Extreme points of the class $Q^{\beta}(\alpha,\tau,\gamma)$

We begin with the statement and the proof of the following results.

2.1. Theorem 2.1

A function $f(z)^{\beta} \in Q^{\beta}(\alpha,\tau,\gamma)$ if and only if $f(z)^{\beta}$ can be expressed as

$$\mathrm{f}(z)^{\beta} = \frac{1}{\alpha+\tau}\int_{|x|=1}\left[(2\gamma-\alpha-\tau)\,z^{\beta} + 2\,(\alpha+\tau-\gamma)\sum_{\kappa=0}^{\infty}\frac{(\alpha+\tau)\,\beta\ x^{\kappa}\ z^{\beta+\kappa}}{\alpha\ \beta+\tau\,(\kappa+\beta)}\right]d\mu\,(x) \qquad ..2.1$$

where $\mu(x)$ is the probability measure defined on X = $\{x : |x| = 1\}$. *For fixed α, τ, β and γ*; $Q^{\beta}(\alpha, \tau, \gamma)$ *and the probability measure μ defined on* X *are one-to-one by expression (2.1).*

Proof. By the definition of $Q^{\beta}(\alpha,\tau,\gamma)$, we know that $f(z)^{\beta} \in Q^{\beta}(\alpha,\tau,\gamma)$ if and only if

$$\frac{\tau\left(f(z)^{\beta}\right)'\left(z^{\beta}\right)' + \alpha f(z)^{\beta}/z^{\beta} - \gamma}{\alpha+\tau-\gamma} \in \mathrm{P} \qquad \text{………………..2.2}$$

by Lemma 1.2

$$\frac{\tau\left(f(z)^{\beta}\right)'\left(z^{\beta}\right)' + \alpha f(z)^{\beta}/z^{\beta} - \gamma}{\alpha+\tau-\gamma} = \int_{|x|=1}\frac{1+xz}{1-xz}d\mu\,(x) \qquad \text{………………2.3}$$

and (2.3) can be written as

$$\frac{\left(f(z)^{\beta}\right)'}{\left(z^{\beta}\right)'} + \frac{\alpha}{\tau}\frac{f(z)^{\beta}}{z^{\beta}} = \frac{1}{\tau}\int_{|x|=1}\frac{(\alpha+\tau)+(\alpha+\tau-2\gamma)\,xz}{1-xz}d\mu\,(x) \qquad \text{…………..2.4}$$

Which yields

$$z^{\frac{-\alpha\beta-\tau\beta+\tau}{\tau}} \int_0^z \left[\frac{\left(f(\xi)^{\beta}\right)'}{(\xi^{\beta})'} + \frac{\alpha}{\tau}\frac{f(\xi^{\beta})}{\xi^{\beta}}\right] \xi^{\frac{\alpha\beta+\tau\beta-\tau}{\tau}} d\xi$$

……………..2.5

$$= \frac{1}{\tau}\int_{|x|=1}\left[z^{\frac{-\alpha\beta-\tau\beta+\tau}{\tau}} \int_0^z \frac{(\alpha+\tau)+(\alpha+\tau-2\gamma)\,x\xi}{1-x\xi}\xi^{\frac{\alpha\beta+\tau\beta-\tau}{\tau}} d\xi\right] d\mu\,(x$$

……………2.6

and so

$$f(\beta) = \frac{1}{\alpha+\tau}\int^{|x|=1}\left[(2\gamma-\alpha-\tau)\,z^{\beta} + 2\,(\alpha+\tau-\gamma)\sum_{\kappa=0}^{\infty}\frac{(\alpha+\tau)\,\beta\;x^{\kappa}z^{\beta+\kappa}}{\alpha\beta+\tau\,(\beta+\kappa)}\right] d\mu\,(x)\,.$$

……2.7

The converse follows, from ([7], page 288), we know that both probability measures μ and class P; class P and $Q^{\beta}(\alpha,\tau,\gamma)$ are one-to-one and this is the second part of the theorem.*2*

2.2. Corollary 2.2

The extreme points of the class $Q^{\beta}(\alpha,\tau,\gamma)$ are

$$f_x\,(z)^{\beta} = \frac{1}{\alpha+\tau}\left[(2\gamma-\alpha-\tau)\,z^{\beta} + 2\,(\alpha+\tau-\gamma)\sum_{\kappa=0}^{\infty}\frac{(\alpha+\tau)\,\beta\;x^{\kappa}z^{\beta+\kappa}}{\alpha\beta+(\beta+\kappa)\,\tau}\right] \quad \text{………(2.8)}$$

Proof. Using the notation $f_x(z)$, (2.8) can be written as

$$f_{\mu}(z) = \int_{|x|=1} f_x(z)\; d\mu(x). \quad \text{…………………… (2.9)}$$

By Theorem 2.1, the map $\mu \to f_{\mu}$ is one-to-one, so the assertion follows from ([7],page 288).

2.3. Corollary 2.3

If $f(z)^{\beta} = z^{\beta} + \sum_{\kappa=2}^{\infty} a_{\kappa} z^{\kappa} \in Q^{\beta}(\alpha,\tau,\gamma)$ *then for $\kappa \geq 2$, we have*

$$|a_{\kappa}| \leq \frac{2\,(\alpha+\beta-\gamma)}{\alpha\beta+(\beta+\kappa-1)\,\tau} \quad \text{……………..(2.10)}$$

Proof. From (2.8)

$$f_x\,(z)^{\beta} = z^{\beta} + 2\,(\alpha+\tau-\gamma)\sum_{k=2}^{\infty}\frac{\beta\;x^{\kappa-1}z^{\beta+\kappa-1}}{\alpha\beta+\tau\,(\beta+\kappa-1)} \quad (|x| = 1)$$

…………2.11

Comparing the coefficient yields the result*2*

2.4. Corollary 2.4

If $f(z)^{\beta} \in Q^{\beta}(\alpha, \tau, \gamma)$, then for $|z| = r < 1$, we have

$$\left|f(z)^{\beta}\right| \leq r^{\beta}+2(\alpha+\tau-\gamma) \sum_{\kappa=2}^{\infty} \frac{r^{\beta+\kappa-1}}{\alpha \beta+(\beta+\kappa-1) \tau} \quad \ldots\ldots\ldots\ldots\ldots(2.12)$$

Proof. The result follows from (2.11).

3. Radius of univalency

Now we calculate the radius of univalency for functions of the class $Q^{\beta}(\alpha, \tau, \gamma)$

3.1. Theorem 3.1

Let $f(z)^{\beta} \in Q^{\beta}(\alpha, \tau, \gamma)$, *then* $f(z)^{\beta}$ *is univalent in* $|z| < R^{\beta}(\alpha, \tau, \gamma)$, *where*

$$\mathrm{R}(\alpha, \tau, \gamma)=\inf_{\beta+\kappa-1}\left\{\frac{\alpha \beta+(\beta+\kappa-1) \tau}{2(\beta+\kappa-1)(\alpha+\tau-\gamma)}\right\}^{\frac{1}{\kappa-1}} \quad \ldots\ldots\ldots 3.1$$

Proof. It suffices to show that

$$\left|\frac{(f(z)^{\beta})'}{(z^{\beta})'}-1\right|<1$$

that is

$$\left|\frac{((z)^{\beta})'}{(z^{\beta})'}-1\right|=\left|\sum_{\kappa=2}^{\infty}(\beta, \kappa,-1) a_{\kappa} z^{\kappa-1}\right|$$

$$<\sum_{\kappa=2}^{\infty}(\beta+\kappa-1)\left|a_{\kappa}\right||z|^{\kappa-1}.$$

This is less than 1 if

$$|z|^{\kappa-1}<\frac{\alpha \beta+(\beta+\kappa-1) \tau}{2(\alpha+\tau-\gamma)(\beta+\kappa-1)}.$$

4. Conclusion

This research is a generalization of the work of Zhi-Gang and it exterblished that $Q(\alpha,\tau,\gamma)$ in is equivalent to $Q^{1}(\alpha, \tau, \gamma)$.

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