International Journal of Science and Research Archive
eISSN: 2582-8185
Cross Ref DOI: 10.30574/ijsra
Journal homepage: https://ijsra.net/

(RESEARCH ARTICLE)

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Comparative study of Symmetric Gauss-Seidel methods and preconditioned Symmetric Gauss-Seidel methods for linear system

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International Journal of Science and Research Archive, 2023, 08(01), 940–947

Publication history: Received on 01 January 2023; revised on 14 February 2023; accepted on 17 February 2023

Article DOI: https://doi.org/10.30574/ijsra.2023.8.1.0155

Abstract

This paper deals with the comparative study of preconditioned Symmetric Gauss-Seidel (SGS), New Symmetric Gauss-Seidel (NSGS), and Parametric Symmetric Gauss-Seidel (PSGS) methods for solving the linear system Ax = b are considered. This system is preconditioned with precondition type I + S. Convergence properties are analyzed with standard procedures and a numerical experiment is undertaken to compare the efficiency of the matrix. Algorithms are prepared. MATLAB software is used for checking computational efficiency of preconditioned iterative methods. Results indicate the effectiveness of preconditioning.

Keywords: Symmetric Gauss-Seidel (SGS) method; New Symmetric Gauss-Seidel (NSGS) method; Parametric Symmetric Gauss-Seidel (PSGS) method; Condition number; Spectral radius

1. Introduction

In computational mathematics, an iterative method is a mathematical procedure that uses an initial value to generate a sequence of improving approximate solutions for a class of problems, in which the k^{th} approximation includes the termination criteria [1]. Classical iterative methods for solving linear systems based on converting the system $Ax = b$, where $A \epsilon R^{n\times n}$, $b \epsilon R^{n\times 1}$ are given and $x \epsilon R^{n}$ is unknown, into the system $x = Cx + d$ and generating a sequence of approximation

$$x^{(k+1)} = Cx^{(k)} + d.$$

The stationary iterative methods to solve a system of linear equations considered here are as follows [2]:

The Gauss-Seidel method of solving a linear system is a simple iterative technique obtained by transforming the linear system:

$$Ax = b \text{ ..... (1)}$$

Into the system $= Cx + d$. Let the splitting of matrix A as

$$A = D + L + U \text{ ..... (2)}$$

Where D, L, and U represent the diagonal, lower triangular, and upper triangular parts of the coefficient matrix A, respectively. Then the equation to be solved for (1) can be rewritten as:

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$$(D + L + U)x = b$$

$$\text{Or, } (D + L)x = -Ux + b$$

$$\text{Or, } x = (D + L)^{-1}(-Ux + b)$$

Thus, the iterative Gauss-Seidel method is as:

$$x^{(k+1)} = (D + L)^{-1}\left(-Ux^{(k)} + b\right) \text{ ..... (3)}$$

This method is also referred to as the forward Gauss-Seidel method. There is a method known as the backward Gauss-Seidel method. This method is defined as:

The equation to be solved for (1) can be rewritten as:

$$(D + U)x = -Lx + b$$

$$\text{Or, } x = (D + U)^{-1}(-Lx + b)$$

Thus, the iterative backward Gauss-Seidel method is as

$$x^{(k+1)} = (D + U)^{-1}\left(-Lx^{(k)} + b\right) \text{ ..... (4)}$$

The Symmetric Gauss-Seidel (SGS) method is obtained by combining the forward and backward variants of the Gauss-Seidel method. Thus, the iterative SGS method from (3) and (4) is defined

$$x^{(k+1)} = (D + U)^{-1}L(D + L)^{-1}Ux^{(k)} + (D + U)^{-1}[I - L(D + L)^{-1}]b \text{ ..... (5)}$$

Also, the New Symmetric Gauss-Seidel (NSGS) method from the forward and backward variants after reversing the order is:

$$x^{(k+1)} = (D + L)^{-1}U(D + U)^{-1}Lx^{(k)} + (D + L)^{-1}[I - U(D + U)^{-1}]b \text{ ..... (6)}$$

Considering linear combination of equations (3) and (4), a new method is deduced, where the parameter $\mu \in [0,1]$. is:

$$x^{(k+1)} = -[\mu(D + L)^{-1}U + (1 - \mu)(D + U)^{-1}L]x^{(k)} + [\mu(D + L)^{-1} + (1 - \mu)(D + U)^{-1}]b \text{ ..... (7)}$$

Called Parametric Symmetric Gauss-Seidel (PSGS) method [3].

1.1. Review of literature

In this study, we show that when certain iterative approaches are used to certain preconditioned systems, the results are faster than when those same methods are applied to the same original system under specific assumptions. The various research for iterative methods can be found as [4] performed certain elementary row operations on A before applying the Gauss-Seidel or Jacobi iterative methods. It is concluded that when A is a non-singular M-matrix or a singular tridiagonal M-matrix, the modified method yields considerable improvement in the rate of convergence for the iterative method. The study showed that this method is superior to certain other modified iterative methods. The performance of this modified method on some matrices other than M-matrices is also investigated. [5] developed the modified Jacobi and Gauss- Seidel methods for Z-matrices. The modified proved to be the effective method for Z-matrix. [6] examined several preconditioned iterative methods reported in the literature that have been used for improving the convergence rate of the Gauss-Seidel preconditioners, they found that the preconditioner of the Gauss-Seidel method is more effective than the preconditioner of the Accelerated Overrelaxation method. [7] demonstrated comparison results for preconditioned Gauss-Seidel methods. A new strict comparison result between two different preconditioned Gauss-Seidel methods was developed, some errors were pointed out and a new proof for the corresponding results was presented. [8] obtained the best iterative solutions and preconditioning methods for major kinds of large and sparse linear systems. Scientific computing simulations take longest to solve large and sparse linear systems. More data improves mathematical models, but it involves solving complex algebraic equations. Recently, iterative solvers have been employed to efficiently handle large sparse and/or structured systems created through numerical model

discretization. [9] begins with Gauss who developed the first known method that can be termed iterative. The early 20th century saw good progress of these methods which were initially used to solve least-squares systems and then linear systems arising from the discretization of partial differential equations. [10] studied the convergence of Gauss-Seidel method and compare the convergence with preconditioned version of Gauss-Seidel method. [11] investigated unsaturated flow problems using Richard's equation. After linearizing Richard's equation, the finite difference method is utilised for numerical discretization and iterative calculation. When the discrete space step is tiny and the time step is big, Jacobi, Gauss-Seidel, and Successive Overrelaxation iterative algorithms converge slowly. Therefore, they used the multistep preconditioner to improve traditional iterative methods and proposed an improved Gauss-Seidel iterative method (ICMP(m)-GS) with a multistep preconditioner based on the integral correction method to solve the linear equations derived from the linearized Richard's equation. The results indicated that the suggested ICMP(m)-GS considerably improves linear equations. ICMP(m)-GS outperforms traditional methods and a single improvement method in convergence rate, calculation efficiency, and accuracy.

1.2. Preconditioned iterative methods

Preconditioning is a method of applying a transformation, known as the precondition, to the original linear system to produce one with the same solution but a better-suited for numerical solution.

Let us now examine the preconditioned linear system

$$\hat{A}x = \hat{b} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Take into account the precondition by [12] in order to solve the preconditioned system (8) where

$$\hat{A} = (I + S)A \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

And

$$\hat{b} = (I + S)b \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

With

$$S = \begin{bmatrix} 0 & 0 & \cdots & -a_{1n} \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Without loss of generality, let the matrix $\hat{A}$ be

$$\hat{A} = \hat{D} + \hat{L} + \hat{U}$$

In this case, the diagonal matrix is designated by $\hat{D}$, and the strictly lower and upper triangular matrices produced from $\hat{A}$ are denoted by $\hat{L}$ and $\hat{U}$, respectively. The technique will converge if matrix $\hat{A}$ has a diagonally dominant position, as is well known. This is a prerequisite that must be fulfilled, but it is not necessary. Following is a definition of the preconditioned forward Gauss-Seidel method:

$$x^{(k+1)} = -\left(\hat{D} + \hat{L}\right)^{-1}\hat{U}x^{(k)} + \left(\hat{D} + \hat{L}\right)^{-1}\hat{b} \quad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

And the preconditioned backward Gauss-Seidel method as:

$$x^{(k+1)} = -\left(\hat{D} + \hat{U}\right)^{-1}\hat{L}x^{(k)} + \left(\hat{D} + \hat{U}\right)^{-1}\hat{b} \quad \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

In the same way the preconditioned Symmetric Gauss-Seidel (SGS) method is:

$$x^{(k+1)} = \left(\hat{D} + \hat{U}\right)^{-1}\hat{L}\left(\hat{D} + \hat{L}\right)^{-1}\hat{U}x^{(k)} + \left(\hat{D} + \hat{U}\right)^{-1}\left[I - \hat{L}\left(\hat{D} + \hat{L}\right)^{-1}\right]\hat{b} \quad \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

the preconditioned New Symmetric Gauss-Seidel (NSGS) method as:

$$x^{(k+1)} = \left(\widehat{D} + \hat{L}\right)^{-1}\widehat{U}\left(\widehat{D} + \widehat{U}\right)^{-1}\hat{L}x^{(k)} + \left(\widehat{D} + \hat{L}\right)^{-1}\left[I - \widehat{U}\left(\widehat{D} + \widehat{U}\right)^{-1}\right]\hat{b} \text{ ..... (14)}$$

the preconditioned Parametric Symmetric Gauss-Seidel (PSGS) method as

$$x^{(k+1)} = -\left[\mu\left(\widehat{D} + \hat{L}\right)^{-1}\widehat{U} + (1-\mu)\left(\widehat{D} + \widehat{U}\right)^{-1}\hat{L}\right]x^{(k)} + \left[\mu\left(\widehat{D} + \hat{L}\right)^{-1} + (1-\mu)\left(\widehat{D} + \widehat{U}\right)^{-1}\right]\hat{b} \text{ ....(15)}$$

1.3. Operational terms

In this paper following operational terms are used

1.3.1. Spectral radius

Spectral radius is the absolute value of the greatest eigenvalues of a square matrix A is denoted by $\rho(A) = max_{\lambda_i \epsilon A}|\lambda_i|, 1 \leq i \leq n$.

The spectral radius of the iterative matrix is conclusive for the convergence and stability of the method and the smaller it is, the faster the method converges.

1.3.2. Condition number

The condition number of a square matrix A is defined by $\kappa(A) = \|A\|.\|A^{-1}\|$.

The condition number of singular matrix is ∞.

Objectives

The objectives of this paper are to

- Develop algorithms for SGS, NSGS, PSGS and preconditioned SGS, NSGS, PSGS methods for linear systems
- Evaluate SGS, NSGS, PSGS and preconditioned SGS, NSGS, PSGS methods for existing preconditioned linear systems.
- Compare the convergence, spectral radius, iterations count, condition number, and tolerance for SGS, NSGS, PSGS and preconditioned SGS, NSGS, PSGS methods.

2. Material and methods

In this study the convergence of SGS, NSGS, PSGS and preconditioned SGS, PSGS, NSGS methods was tested by using the coefficient matrix of the linear system of equations. The results were obtained by

- Using an initial approximation for iterative methods.
- Judging the convergence of successive approximation and exact solution.
- Developing the algorithmic approach for the results.
- Matlab software.
- Validating scheme using numeric experiment.

2.1. Numerical experiment

Considering example 4.2 of [13], the matrix selected for the experiment is as follows:

$$A = \begin{bmatrix} 3 & 0 & 4 \\ 7 & 4 & 2 \\ -1 & 1 & 2 \end{bmatrix}$$

The vector b was selected in this case so that it yields the same initial guess of $x^0 = 0$ and the exact answer of $x_i = i,\ \forall\ i = 1,2, \dots, n$. The stooping criteria were

$$max_{1 \leq i \leq n}\left|x_i^{(k+1)} - x_i^{(k)}\right| < 10^{-14}$$

2.2. Criteria for effectiveness

The criteria under discussion are as follows:

- When $\rho(C) < 1$ and approximately equal to 1, $\rho(C) - \rho(\hat{C})$ seems to be relatively small, then improvement seems to be rather slight where $\hat{C}$ is the preconditioned iteration matrix.
- When $\rho(C) < 1$ and close to 0.5, $\rho(C) - \rho(\hat{C})$ slightly large, then modified method will be preferred over standard method where $\hat{C}$ is the preconditioned iteration matrix.

2.3. Algorithms

Following algorithms were developed for:

2.3.1. Algorithm for SGS method

- Enter the matrix A, b and choose an initial guess x^0, maximum number of iterations, and tolerance (ϵ).
- Find D, L, and U of the matrix A.
- Find $(D+U)^{-1}, (D+L)^{-1}$
- Set $C = (D+U)^{-1}L(D+L)^{-1}U$
- Set d $= (D+U)^{-1}[I - L(D+L)^{-1}]b$
- Compute $x^{(k+1)} = Cx^{(k)} + d$
- Stop if $max_{1\le i\le n}\left|x_i^{(k+1)} - x_i^{(k)}\right| < \epsilon$

2.3.2. Algorithm for preconditioned SGS method

- Enter the matrix $\hat{A}$, $\hat{b}$ and choose an initial guess x^0, maximum number of iterations, and tolerance (ϵ).
- Find $\hat{D}$, $\hat{L}$, and $\hat{U}$ of the matrix $\hat{A}$.
- Find $\left(\hat{D}+\hat{L}\right)^{-1}, \left(\hat{D}+\hat{U}\right)^{-1}$
- Set $\hat{C} = \left(\hat{D}+\hat{U}\right)^{-1}\hat{L}\left(\hat{D}+\hat{L}\right)^{-1}\hat{U}$
- Set $\hat{d} = \left(\hat{D}+\hat{U}\right)^{-1}\left[I - \hat{L}\left(\hat{D}+\hat{L}\right)^{-1}\right]\hat{b}$
- Compute $x^{(k+1)} = \hat{C}x^{(k)} + \hat{d}$
- Stop if $max_{1\le i\le n}\left|x_i^{(k+1)} - x_i^{(k)}\right| < \epsilon$

2.3.3. Algorithm for NSGS method

- Enter the matrix A, b and choose an initial guess x^0, maximum number of iterations, and tolerance (ϵ).
- Find D, L, and U of the matrix A.
- Find $(D+U)^{-1}, (D+L)^{-1}$
- Set $C = (D+L)^{-1}U(D+U)^{-1}$
- Set d $= (D+L)^{-1}[I - U(D+U)^{-1}]b$
- Compute $x^{(k+1)} = Cx^{(k)} + d$
- Stop if $max_{1\le i\le n}\left|x_i^{(k+1)} - x_i^{(k)}\right| < \epsilon$

2.3.4. Algorithm for preconditioned NSGS method

- Enter the matrix $\hat{A}$, $\hat{b}$ and choose an initial guess x^0, maximum number of iterations, and tolerance (ϵ).
- Find $\hat{D}$, $\hat{L}$ and $\hat{U}$ of the matrix $\hat{A}$.
- Find $\left(\hat{D}+\hat{L}\right)^{-1}, \left(\hat{D}+\hat{U}\right)^{-1}$
- Set $\hat{C} = \left(\hat{D}+\hat{L}\right)^{-1}\hat{U}\left(\hat{D}+\hat{U}\right)^{-1}\hat{L}$
- Set $\hat{d} = \left(\hat{D}+\hat{L}\right)^{-1}\left[I - \hat{U}\left(\hat{D}+\hat{U}\right)^{-1}\right]\hat{b}$
- Compute $x^{(k+1)} = \hat{C}x^{(k)} + \hat{d}$
- Stop if $max_{1\le i\le n}\left|x_i^{(k+1)} - x_i^{(k)}\right| < \epsilon$

2.3.5. Algorithm for PSGS method

- Enter the matrix A, b and choose an initial guess x^0, maximum number of iterations, $\mu\epsilon[0,1]$ and tolerance (ϵ).

- Find D, L, and U of the matrix A.
- Find $(D+U)^{-1}, (D+L)^{-1}$
- Set C $= -[\mu(D+L)^{-1}U + (1-\mu)(D+U)^{-1}L]$
- Set d $= [\mu(D+L)^{-1} + (1-\mu)(D+U)^{-1}]b$
- Compute $x^{(k+1)} = Cx^{(k)} + d$
- Stop if $max_{1\le i\le n}\left|x_i^{(k+1)} - x_i^{(k)}\right| < \epsilon$

2.3.6. Algorithm for preconditioned PSGS method

- Enter the matrix $\hat{A}$, $\hat{b}$ and choose an initial guess x^0, maximum number of iterations, tolerance (ϵ),and $\mu\epsilon[0,1]$.
- Find $\widehat{D}$, $\hat{L}$ and $\widehat{U}$ of the matrix $\hat{A}$.
- Find $\left(\widehat{D}+\hat{L}\right)^{-1}, \left(\widehat{D}+\widehat{U}\right)^{-1}$
- Set $\hat{C} = -\left[\mu\left(\widehat{D}+\hat{L}\right)^{-1}\widehat{U} + (1-\mu)\left(\widehat{D}+\widehat{U}\right)^{-1}\hat{L}\right]$
- Set $\hat{d} = \left[\mu\left(\widehat{D}+\hat{L}\right)^{-1} + (1-\mu)\left(\widehat{D}+\widehat{U}\right)^{-1}\right]\hat{b}$
- Compute $x^{(k+1)} = \hat{C}x^{(k)} + \hat{d}$
- Stop if $max_{1\le i\le n}\left|x_i^{(k+1)} - x_i^{(k)}\right| < \epsilon$

3. Results and discussion

Computational efficiency of preconditioned SGS, NSGS, PSGS method is enhanced by using MATLAB software, as noted in [14]. For each approach, following tables show the condition number, determinant, spectral radius and iterations:

Table 1 SGS method

Matrix A	Condition Number	Determinant	Spectral Radius	Iterations
Without precondition	∞	0	1.5833	Divergent
With precondition	∞	0	0.7095	102

Table 1 clearly indicates that SGS method is divergent whereas preconditioned SGS method is convergent.

Table 2 NSGS method

Matrix A	Condition Number	Determinant	Spectral Radius	Iterations
Without precondition	∞	0	1.5833	Divergent
With precondition	∞	0	0.7095	100

Table 2 points out that the preconditioned NSGS method is convergent and NSGS method is divergent.

Table 3 PSGS method

μ	Condition Number	Determinant	Spectral Radius	Iterations
0.1	9.9688	-0.2441	1.0380	Divergent
0.2	5.0096	-0.3858	1.0017	Divergent
0.3	4.2469	-0.4430	0.9523	671
0.4	4.4840	-0.4340	0.8814	268
0.5	5.448	-0.3767	0.7843	141
0.6	7.4662	-0.2893	0.6770	91
0.7	11.2000	-0.1899	0.8119	169

0.8	18.9087	-0.0964	1.0514	Divergent
0.9	42.2889	-0.0271	1.3205	Divergent

Table 3 shows that the PSGS method converges for some values of parameter μ and diverges for some values of μ.

Table 4 Preconditioned PSGS method

μ	Condition number	Determinant	Spectral Radius	Iterations
0.1	25.7938	0.0595	0.8879	284
0.2	10.8862	0.0944	0.6566	80
0.3	9.3984	0.1089	0.4876	49
0.4	10.4580	0.17074	0.5027	51
0.5	12.6310	0.0941	0.4975	51
0.6	17.0651	0.0732	0.4766	48
0.7	25.8323	0.0491	0.4437	44
0.8	44.7167	0.0260	0.4059	40
0.9	110.6320	0.0082	0.3817	37

Table 4 displays that the preconditioned PSGS method converges for every value of the parameter μ.

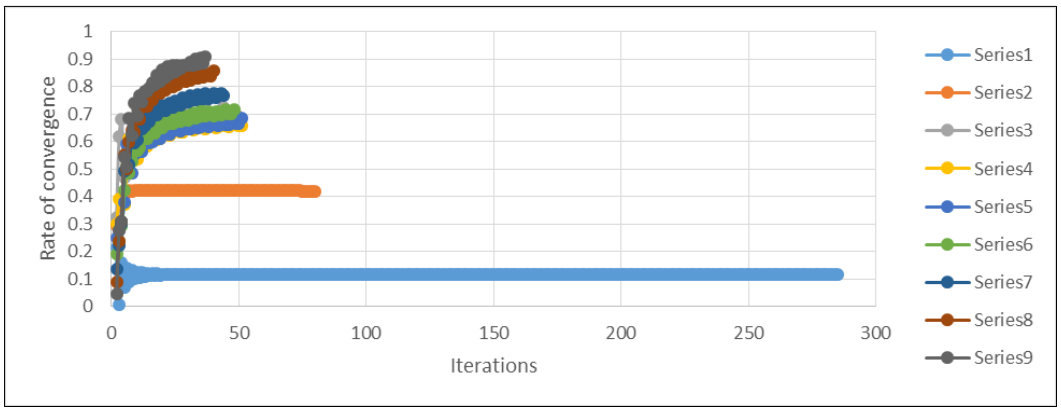

Figure 1 Graphical representation of relationship between rate of convergence and iterations

Figure 1 indicates that as the number of iterations decrease for the values of parameter from 0.1 to 0.9 for preconditioned PSGS method, the rate of convergence increases correspondingly. For the value of the parameter 0.9 we obtained the least number of iterations and maximum convergence rate. As higher convergence rate shows the effectiveness of the preconditioned PSGS method.

4. Conclusion

The above study reveals that Symmetric Gauss-Seidel (SGS), New Symmetric Gauss-Seidel (NSGS) and Parametric Symmetric Gauss-Seidel (PSGS) methods are divergent. Whereas preconditioned SGS, NSGS and PSGS methods are convergent for the numeric experiment under consideration. Also, the preconditioned PSGS method have least spectral radius and maximum rate of convergence.

Compliance with ethical standards

Acknowledgments

The authors would like to thank the Department of Mathematics, Kurukshetra University, Kurukshetra. Niketa is thankful to CSIR for providing JRF.

Disclosure of conflict of interest

The authors declare that we have no conflict of interests.

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