International Journal of Science and Research Archive

eISSN: 2582-8185

Cross Ref DOI: 10.30574/ijsra

Journal homepage: https://ijsra.net/

(Review Article)

Check for updates

On τ−valently starlike and τ−valently close-to-convex of q-operator

Uzoamaka Azuka Ezeafulukwe *

Department of Mathematics, Faculty of Physical Sciences University of Nigeria, Nsukka, Nigeria.

International Journal of Science and Research Archive, 2023, 08(02), 716–723

Publication history: Received on 03 February 2023; revised on 14 March 2023; accepted on 17 March 2023

Article DOI: https://doi.org/10.30574/ijsra.2023.8.2.0068

Abstract

In this article, some properties of τ−valently starlike and τ−valently close-toconvex of q−operator are studied.

Keywords: τ−valently function; Starlike function; Close-to-convex function; q-Operator

1. Introduction

Let A(τ) denote the class functions which are analytic and τ−valent in a unit disc U = {z : z ∈ C ,|z| <1} of the form

$$f(z) = z^{\tau} + \sum_{\kappa=1}^{\infty} \tau^{\kappa} a_{\kappa} z^{\tau+\kappa} \quad \tau \in \mathbb{N}, z \neq \frac{1}{\tau}, \tau|z| < 1 \qquad \text{.....................(1.1)}$$

The function $f(z)$ ∈ A(τ) was defined in [1] as

$$\Theta(z) * \frac{z^{\tau}}{1-\tau z}, \quad z \neq \frac{1}{\tau}, \tau|z| < 1, \tau \in \mathbb{N},$$

where $\Theta(z) = \sum_{\kappa=0}^{\infty} a_{\kappa} z^{\tau+\kappa}, \ a_0 = 1.$

Let functions $\varrho(z) \prec \sigma(z)$ and σ be analytic in U, then ϱ is said to be subordinate to σ, written as $\varrho(z) \prec \sigma(z)$, if $\varrho(0) = \sigma(0)$ with $\sigma(z)$ univalent in U and the set of values assumed by $\varrho(z)$ for z ∈ U is included in the set of values assumed there by $\sigma(z)$.

A function $f(z)$ in $A(\tau)$ is said to be τ−valently starlike if and only if

$$\Re e \left\{ \frac{z f'(z)}{f(z)} \right\} > 0$$

1 ,.....................(1.2)

also function $f(z)$ in $A(\tau)$ is said to be τ−valently close-to-convex if and if there exists a starlike function say $g(z)$ such that

* Corresponding author: Uzoamaka A. Ezeafulukwe

$$\Re e\left\{\frac{zf'(z)}{g(z)}\right\} > 0 \quad \ldots\ldots\ldots\ldots\ldots(1.3)$$

We denote the class of functions which are τ−valently starlike and τ−valently close-to convex by S(τ) and K(τ) respectively. Some properties of functions in classes S(τ) and K(τ) had been studied extensively for example see (pages 188-198, [11]) and also [2].

Definition 1.1 *Let $\mu \in$ C be fixed. A set $\eta \subseteq$ C is called a μ−geometric set if for $z \in \eta$, $\mu z \in \eta$. Let $h \in$ A(1),0 < q <1 be a function defined on a q-geometric set $\eta \in$ C. The q-difference operator is defined by the formula*

$$\mathcal{D}_q h(z) = \frac{h(z) - h(qz)}{z - qz}, \quad z \in \eta - \{0\} \quad ,\ldots\ldots\ldots\ldots\ldots\ldots(1.4)$$

and $D_q h(0) = h'(0)$. *From (1.4), we have*

$$\mathcal{D}_q h(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}, \ (z \neq 0)$$

where

$$[k]_q = \frac{1-q^k}{1-q} .$$

As $q \to 1$, $[k]_q \to k$. In particular when $h(z) = z^k$, clearly $\mathcal{D}_q \to \frac{d}{dq}$ $as\ q \to 1$.

The q-shift factorial, the multiple q-shift factorial and the q-binomial coefficients are defined by

$$(a_1, a_2, \cdots, a_n ;q)_k = \begin{cases} 1 & (k = 0, j = 1, a_1 = a) \\ \Pi_{n=0}^{k-1}(1 - aq^n) & (j = 1, k \neq 0, a_1 = a, n \in \mathbb{N},) \\ \Pi_{j=1}^{n}(a_j ;q) & (j = 1,2,\cdots,n; n \in \mathbb{N}, k \in \mathbb{Z}). \end{cases}$$

………..1.5

and

$$\begin{bmatrix} a \\ k \end{bmatrix}_q = \begin{cases} 1 & (k = 0), \\ \frac{(1-q^a)(1-q^{a-1})\cdots(1-q^{a-k+1})}{(q;q)_k} & (k \in \mathbb{N}) \end{cases}$$

………..1.6

where $a, q \in$ C.

Let ${}_r\Phi_s$ denote the q-Hypergeometric series

$${}_r\Phi_s\begin{pmatrix} a_1, & \cdots, & a_r & \Big| \\ & \vdots & & \Big| q,z \\ b_1, & \cdots, & b_s & \Big| \end{pmatrix} = {}_r\Phi s(a_1,\ldots,a_r ; b_1,\cdots,b_s;q,z)$$

……………..1.7

$$= \begin{bmatrix} a \\ 0 \end{bmatrix}_q + \sum_{k=1}^{\infty} \frac{(a_1, \ldots, a_r; q)_k}{(q, b_1, \ldots, b_s; q)_k} z^k (-q^{\frac{(k-1)}{2}})^{k(s+1-r)}$$ ………………1.8

For more details about q-derivatives see [14] and also [3]-[5].

Let $s + 1 = r$; $r, s \in N0 = N \cup \{0\}$ then, the function

$_r G_s(a_1, \ldots, a_r; b_1, \ldots, b_s, q, z^\tau) = z^\tau {}_r\Phi_s(a_1, \ldots, a_r; b_1, \ldots, b_s, q, z)$

$$= z^\tau + \sum_{k=1}^{\infty} \frac{(a_1, q)_k \ldots (a_r, q)_k}{(q, q)_k (b_1, q)_k \ldots (b_s, q)_k} z^{\tau+k}$$ …………………………(1.9)

In this article we assume

$$M^{(n)}(a_1,\ldots,a_r;b_1,\ldots,b_s; q, z^\tau)f(z) = M^{(n)}(a_i ; b_j ;q , z^\tau)f(z),$$

where ($n \in N$; $i = 1,2,\ldots,r$; $j = 1,2,\ldots,s$).

Let $0 < \lambda < 1$ we define the operator $M^{(n)}(a_i ; b_j ; q , z^\tau)f(z) : A(\tau) \to A(\tau)$ by

$$\mathcal{M}^{(n)}(a_i; b_j; q, z^\tau)f(z) = \mathcal{M}^{(1)}\left(\mathcal{M}^{(n-1)}(a_i; b_j; q, z^\tau)f(z)\right)$$ ……………………..(1.10)

$$= z^\tau + \sum_{k=2}^{\infty}[1 + \lambda(k-1)]^n \tau^{k-1}\theta_{k-1}\, a_k z^{r+k-1},$$ ……………………..(1.11)

where

$$\theta_{k-1} = \frac{(a_1,q)_{k-1} \ldots (a_{r,q})_{k-1}}{(q,q)_{k-1}(b_1,q)_{k-1} \ldots (b_{r,q})_{k-1}}$$

Clearly

$$M^{(0)}(a_i ; b_j ; q , z^\tau)f(z) = {}_r G_s(a_I ; b_j ; q , z^\tau) * f(z),$$

and

$$M^{(n)}(a_i; b_j; q, z^\tau)f(z)$$

$$= (1 - \tau\lambda)\left(\mathcal{M}^{(n-1)}(a_i; b_j; q, z^\tau)f(z)\right) + \lambda z\left[\mathcal{M}^{(n-1)}(a_i; b_j; q, z^\tau)f(z)\right]'$$ …………………(1.12)

The operator $M^{(n)}(a_i;b_j;q,z^\tau)f(z)$ is the generalization of other operator studied in , the item bellow:

- For $r = 1$, $s = 0$, $\tau \in N$, $a_1 = q$, $n = 0$, we have the function $f(z) \in A(\tau)$ studied by many but we mention just a few [8]-[11].
- For $r = 1$, $s = 0$, $\tau = 1$, $a_1 = q$, $n \in N$, $\lambda = 0$ we have the S˘al˘agean operator [13].
- For $r = 1$, $s = 0$, $\tau = 1$ and $a_1 = q$, $n \in N, 0 < \lambda < 1$ we have the Al-Oboudi operator [12].
- For $\tau = 1$, was studied by [3]. **v** For $n = 0$, $\tau = 1$, was studied by [5].
- For $\tau = 1$, $n = 0$, $a_i = q^{\alpha_i}$, $b_j = q^{\beta_j}$, $\alpha_i,\beta_j \in C$, $\beta_j \notin Z^- \cup \{0\}$ ($i = 1,\ldots,r$, $j = 1,\ldots,s$) and $q \to 1$ was studied by [7].

The aim of this article is to calculate some conditions for the operator defined in (1.10) to be τ–valently starlike and τ–valently convex.

2. Criterion for τ−valently starlikeness

In this section, we calculate the necessary condition for the operator defined in (1.10) to be τ−valently starlike. We use the same technique of proof in [6], see also [10] to prove Theorem 2.1

2.1. Theorem 2.1

Assume $z \in \mathrm{U}(\theta) = \{z : |z| < 1, z \neq 0, (\arg z - \theta)(\arg z - \theta - \pi) \neq$

where $\alpha, \theta \in \mathrm{R}$, $0 < \alpha \leq 1$, $0 \leq \theta < \pi$. *Let* $\mathrm{M}^{(n)}(a_i; b_j; q, z^\tau)f(z)$ *be the operator defined in (1.10) such that*

$$\left|\arg \frac{[(\mathcal{M}^{(n+1)}(a_i;b_j;q,z^\tau)f)(z)]}{z^\tau}\right| < \frac{\pi\alpha}{2}, \quad z \in \mathcal{U} \qquad \text{...........2.1}$$

with

$$\left(\mathcal{I}m \frac{[(\mathcal{M}^{(n-1)}(a_i;b_j;q,z^\tau)f)(z)]}{z^\tau}\right)\left(\mathcal{I}m \frac{z}{e^{i\theta}}\right) \neq 0 \qquad \text{,..............(2.2)}$$

then

$$\left|\arg \frac{z[(\mathcal{M}\qquad(a_i;b_j;q,z\quad)f)(z)]}{[(\mathcal{M}^{(n)}(a_i;b_j;q,z^\tau)f)(z)]}\right| < \alpha\pi$$

and $\mathrm{M}^{(n)}(a_i; b_j; q, z^\tau)f(z) \in \mathrm{S}(\tau)$.

Proof. Using the identity (1.12), we set

$$\frac{(\mathcal{M}^{(n)}(a_i;b_j;q,z^\tau)f)(z)}{z(\mathcal{M}^{(n+1)}(a_i;b_j;q,z^\tau)f)(z)}$$

$$= \frac{1}{\lambda\xi}\int_0^1 \left(\frac{(\mathcal{M}^{(n+1)}(a_i;b_j;q,z^\tau)f)(\xi z) - (1-\lambda\tau)(\mathcal{M}^{(n)}(a_i;b_j;q,z^\tau)f)(\xi z)}{z(\mathcal{M}^{(n+1)}(a_i;b_j;q,z^\tau)f)(z)}\right) d\xi$$

$$= \int_0^1 \frac{\left[(\mathcal{M}^{(n)}(a_i;b_j;q,z^\tau)f)(\xi z)\right]'}{(\mathcal{M}^{(n+1)}(a_i;b_j;q,z^\tau)f)(z)} d\xi \qquad \text{........2.3}$$

$$= \frac{z^\tau}{(\mathcal{M}^{(n+1)}(a_i;b_j;q,z^\tau)f)(z)} \int_0^1 \frac{\xi^\tau(\mathcal{M}^{(n)}(a_{i\,;}b_j;\,q,z^\tau)f)(\xi\, z)}{(\xi\, z)^r}$$

and

$$\left|\arg \xi^\tau \frac{(\mathcal{M}^{(n)}(a_i;b_j;q,z^\tau)f)(\xi z)}{(\xi z)^\tau}\right| = \left|\arg \frac{(\mathcal{M}^{(n)}(a_i;b_j;q,z^\tau)f)(\xi z)}{(\xi z)^\tau}\right| < \frac{\pi\alpha}{2} \qquad \text{............2.5}$$

From (2.1), (2.2) and (2.5)we have

- $0 < \arg \left[\frac{(\mathcal{M}^{(n+1)}(a_i;b_j;q,z^\tau)f)(z)}{z^\tau}\right] < \frac{\pi\alpha}{2}$
- $-\frac{\pi\alpha}{2} < \left[\arg \frac{(\mathcal{M}^{(n+1)}(a_i;b_j;q,z^\tau)f)(z)}{z^\tau}\right] < 0$.

If **(i)** holds then, the $\int_0^1 \xi^\tau \frac{[(\mathcal{M}^{(n)}(a_i;b_j;q,z^\tau)f)(\xi z)]}{(\xi z)^\tau} d\xi$ integral lies in the same convex sector $\{z : 0 < \arg z < \frac{\pi\alpha}{2}\}$, and

If **(ii)** holds then, the $\int_0^1 \xi^\tau \frac{[(\mathcal{M}^{(n)}(a_i;b_j;q,z^\tau)f)(\xi z)]}{(\xi z)^\tau} d\xi$ integral lies in the same convex sector $\{z : -\frac{\pi\alpha}{2} < \arg z < 0\}$.

Hence

$$\left|\arg \frac{(\mathcal{M}^{(n)}(a_i;b_j;q,z^\tau)f)(z)}{z(\mathcal{M}^{(n+1)}(a_i;b_j;q,z^\tau)f)(z)}\right|$$

$$\leq \left|\arg \frac{z^\tau}{(\mathcal{M}^{(n+1)}(a_i;b_j;q,z^\tau)f)(z)}\right| + \left|arg \int_0^1 \xi^\tau \frac{(\mathcal{M}^{(n)}(a_i;b_j;q,z^\tau)f)(\xi z)}{(\xi z)^\tau} d\xi\right|$$
$$< \frac{\pi\alpha}{2} + \frac{\pi\alpha}{2} = \pi\alpha,\ z \in \mathcal{U}$$

………………2.6

and the prove of the theory.

We state **Corollaries 2.2-2.4**, without prove because their proves are similar to that of **Theorem 2.1**.

2.2. Corollary 2.2

Let $\mathrm{M}^{(n)}(a_i; b_j; q, z^\tau)f(z)$ *be the operator defined in (1.11) such that*

$$\left|\arg \frac{[(\mathcal{M}^{(n+1)}(a_i;b_j;q,z^\tau)f)(z)]}{z^\tau}\right| < \frac{\pi\alpha}{2},\ z \in \mathcal{U},$$

……………..2.7

If

$$\left(\frac{[(\mathcal{M}^{(n+1)}(a_i;b_j;q,z^\tau)f)(z)]}{z^\tau}\right)$$ …………..2.8

is typically real in U, *then* $(\mathrm{M}^{(n+1)}(a_i; b_j; q, z^\tau)f)(z) \in S(\tau)$ *in* U.

2.3. Corollary 2.3

Let $\tau = 1$ *with* $f(z) \in A(1)$, *and* $\mathrm{M}^{(n)}(a_i; b_j; q, z)f(z)$ *be the operator defined in (1.11) such that*

$$\left|\arg \frac{[(\mathcal{M}^{(n+1)}(a_i;b_j;q,z)f)(z)]}{z}\right| < \frac{\pi\alpha}{2},\ z \in \mathcal{U}$$ ………………2.9

If

$(\mathrm{M}^{(n+1)}(a_i; b_j; q, z)f)(z)$…………………(2.10) *is typically real in* U, *then* $(M^{(n)}(a_i; b_j; q, z)f)(z) \in S(\tau)$ *in* U.

2.4. Corollary 2.4

If $r = 1$, $s = 0$, $\tau \in \mathbb{N}$, $a_1 = q$, $n = 0$ *with*

$J(z) = (M^{(0)}(a_1;b_0;q,z^\tau)f)(z)$ *and* $\left|\arg \frac{z\mathcal{J}'(z)}{z^\tau}\right| < \frac{\pi\alpha}{2}$ *also* $\left(\mathcal{I}m\frac{z\mathcal{J}'(z)}{z^\tau}\right)\left(\mathcal{I}m\frac{z}{e^{i\theta}}\right) \neq 0$ *then*

$$\left|\arg \frac{z\mathcal{J}'(z)}{\mathcal{J}(z)}\right| < \frac{\pi\alpha}{2} \quad \text{,……………..2.11}$$

and $J \in S(\tau)$.

Corollary 2.4 is equivalent to Theorem of Nunokawa ([10],page 24) and the function $J(z)$ can be used to calculate Corollaries 1-3 of [10].

3. Criterion for τ–valently close-to-convex

In this section we calculate the necessary condition for the operator defined in (1.10) to be τ–valently close-to-convex. We use the same technique of proof in [8], see also [9] to prove our **Theorem 3.1**.

3.1. Theorem 3.1

Let $M^{(n)}(a_i; b_j; q, z^\tau)f(z)$ *be the operator defined in (1.10) such that*

$$2 - \frac{2\lambda\tau(\tau-\gamma)}{\tau - d} < 1 + \mathcal{R}e\left\{\frac{(\mathcal{M}^{(n+2)}(a_i;b_j;q,z^\tau)f)(z)}{(\mathcal{M}^{(n+1)}(a_i;b_j;q,z^\tau)f)(z)}\right\} < \gamma, \ (z \in \mathcal{U} \quad \text{……………3.1}$$

where $1 + \tau \leq d$, $\frac{\tau+d-1}{2} < \gamma \leq \frac{\tau+d+1}{2}$ *then* $M^{(n)}(a_i; b_j; q, z^\tau)f(z) \in K(\tau)$.

Proof. For $\lambda = \frac{1}{2}$ and $\tau = 2$, $d = \gamma$, the prove of the theorem is trivial. Let $\lambda \neq \frac{1}{2}$. We set

$$\Theta(z) = \frac{1}{\lambda(\gamma-\tau)}\left\{\lambda(\gamma-\tau) + 1 - \frac{(\mathcal{M}^{(n+2)}(a_i;b_j;q,z^\tau)f)(z)}{(\mathcal{M}^{(n+1)}(a_i;b_j;q,z^\tau)f)(z)}\right\} = \frac{z\vartheta'(z)}{\vartheta(z)} \quad \text{……….(3.2)}$$

From (3.2), we have $\Theta(0) = 1$, $Re\{\Theta(z)\} > 0$ in the unit disc U, it then follows that $\vartheta(z) \in S(1)$, and some calculations on (3.2) using the identity (1.12) gives

$$\frac{(\mathcal{M}^{(n+1)}(a_i;b_j;q,z^\tau)f)(z)}{(z)^\tau} = \left(\frac{\vartheta(z)}{z}\right)^{\tau-\gamma}.$$

Let

$$F(z) = \frac{(\mathcal{M}^{(n+1)}(a_i;b_j;q,z^\tau)f)(z)}{z^{\frac{(\tau+d)}{2}}\vartheta(z)^{\frac{(\tau-d)}{2}}} = \left(\frac{z}{\vartheta(z)}\right)^{\frac{2\gamma-\tau-d}{2}}.$$

$F(z)$ is analytic in U with $F(0) = 1$ so there exist an analytic and univalent function say $s(z)$ such that $s(0) = 1$ and the set of values assumed in $F(z)$ for $z \in$ U is included in the set of values assumed in $s(z)$. We choose such $s(z)$ as $(1 - z)^{2\gamma - d - \tau}$, and hence

$$\mathcal{R}e\left\{\frac{(\mathcal{M}^{(n+1)}(a_i;b_j;q,z^{\tau})f)(z)}{z^{\frac{(\tau+d)}{2}}\vartheta(z)^{\frac{(\tau-d)}{2}}}\right\} > 0 \quad (z \in \mathcal{U}).$$

$$h(z) = z^{\frac{(\tau+d)}{2}}\vartheta(z)^{\frac{(\tau-d)}{2}}, \qquad \text{..................3.3}$$

From (3.3) and the assumption of the theorem, it follows that

$$F(z) \prec s(z) \text{...........} 3.4$$

We set

then

$$\mathcal{R}e\left\{\frac{zh'(z)}{h(z)}\right\} - \frac{\tau+d}{2} + \left(\frac{\tau-d}{2}\right)\mathcal{R}e\left\{\frac{z\vartheta'(z)}{\vartheta(z)}\right\}.$$

From (3.2)and the assumption (3.1) we have

$$\mathcal{R}e\left\{\frac{zh'(z)}{h(z)}\right\} = \frac{\tau+d}{2} + \left(\frac{\tau-d}{2\lambda(\tau-\gamma)}\right)\left\{\lambda(\tau-\gamma) - 1 + \frac{(\mathcal{M}^{(n+2)}(a_i;b_j;q,z^{\tau})f)(z)}{(\mathcal{M}^{(n+1)}(a_i;b_j;q,z^{\tau})f)(z)}\right\}$$

$$> \frac{\tau+d}{2} + \left(\frac{d-\tau}{2\lambda(\tau-\gamma)}\right)\left\{\lambda(\gamma-\tau) + \frac{2d\lambda(\tau-\gamma)}{\tau-d}\right\} = 0 \ (z \in \mathcal{U}) \qquad \text{.........(3.5)}$$

That means $h(z)$ has a zero of order τ at $z = 0$, hence $h(z)$ is τ−valently starlike and from (3.4) and (3.5), $M^{(n)}(a_i;b_j;q,z^{\tau})f(z)$ is τ−valently close-to-convex.

3.2. Corollary 3.2

Let $M^{(n)}(a_i;\, b_j;\, q\,, z^{\tau})f(z)$ *be the operator defined in (1.10) such that*

$$2 - 2\tau\lambda < 1 + \mathcal{R}e\left\{\frac{z(\mathcal{M}^{(n+2)}(a_i;b_j;q,z^{\tau})f)(z)}{(\mathcal{M}^{(n+1)}(a_i;b_j;q,z^{\tau})f)(z)}\right\} < d, \ (z \in \mathcal{U}),$$

or

$$\frac{2\tau\lambda(\gamma-\tau)}{d-\tau} + 1 < 1 + \mathcal{R}e\left\{\frac{z(\mathcal{M}^{(n+2)}(a_i;b_j;q,z^{\tau})f)(z)}{(\mathcal{M}^{(n+1)}(a_i;b_j;q,z^{\tau})f)(z)}\right\} < \gamma, \ (z \in \mathcal{U}).$$

then $M^{(n)}(a_i;\, b_j\,;q\,, z^{\tau})f(z) \in K(\tau)$.

Proof. For $\lambda \neq \frac{1}{2}$, and by substituting $d = \gamma$, or $d = \tau - 2, \ \gamma = \tau - \frac{1}{2}$ in Theorem 3.1, implies the required result.

3.3. Corollary 3.3

Let $J(z)$ *be as Equation (2.9) with* $\tau = 1$ *and* $f(z) \in A(1)$. *Also let* $M^{(0)}(a_i;\, b_j;\, q,\, z)f(z)$ *be the operator defined in (1.11) such that*

$$\frac{2\tau\lambda(\tau-\gamma)}{d-\tau}+\tau<1+\mathcal{R}e\left\{\frac{z\mathcal{J}''(z)}{\mathcal{J}'(z)}\right\}<\gamma,\ (z\in\mathcal{U}) \qquad \text{...............(3.6)}$$

then J(z) ∈ K(τ).

3.4. Corollary 3.4

Let J(z) *be as Equation (2.9) with*

$$\frac{2\tau\lambda(\tau-\gamma)}{d-\tau}+\tau<1+\mathcal{R}e\left\{\frac{z\mathcal{J}''(z)}{\mathcal{J}'(z)}\right\}<\gamma,\ (z\in\mathcal{U}) \qquad \text{.............(3.7)}$$

then J(z) ∈ K(τ).

4. Conclusion

Subclasses of τ−valently starlikeness and τ−valently convex were introduced. The criterion for τ−valently starlikeness and τ−valently convex were studied.

References

[1] K. Al-Shaqsi, M. Darus and O. A. Fadipe-Joseph, A new subclass of SalageanType harmonic Univalent Functions, *Abstract and Applied Analysia*, 2010 (2010), Article ID 821531, 12 pages.

[2] W. Zhi-Gang, G. Chun-Yi and Y. Shao-Mou, On the Univalency of Certain Analytic Functions, *J. Inequal. Pure Appl. Math.*, 7(2006), no .9, pp. 1 - 4.

[3] H. Saitoh, On inequalities of certain analytic functions, *Math. Japon.*, **35** (1990), 1073-1076.

[4] S. Owa, Some properties of certain analytic functions, Soochow J. Math., 13(1987),197-201.

[5] S. Owa, Generalization properties for certain analytic functions, *Internat. J. Math. Math. Sci.*, **21** (1998), 707-712.

[6] D.J. Hallenbeck and T.H. MacGregor, *Linear Problems and Convexity Techniques in Geometric Function Theory*,Pitman Advanced Pub. Program, Boston, London.1984.

[7] D.J. Hallenbeck, Convex hulls and extreme points of some families of univalent functions, *Trans. Amer. Maths. Soc.*, **192** (1975), 285-292.

[8] S.Owa, T. Hayami and K. Kuroki, Some properties of certian analytic functions, *Inter. Jour. Math. and Math. Sci.*, **2007** (2007), 9 pages.