International Journal of Science and Research Archive

eISSN: 2582-8185

Cross Ref DOI: 10.30574/ijsra

Journal homepage: https://ijsra.net/

(Research Article)

Check for updates

Application of some fixed-point theorems in approximation theory

Dr. Sharad Pawar *

Department of Mathematics, S. M. P. Govt. Girls (P.G.) College Madhavpuram, Meerut, Uttar Pradesh, India.

International Journal of Science and Research Archive, 2022, 07(02), 607–613

Publication history: Received on 22 November 2022; revised on 29 December 2022; accepted on 31 December 2022

Article DOI: https://doi.org/10.30574/ijsra.2022.7.2.0367

Abstract

In this paper, we study some fixed point theorems in approximation theory. Fixed point theory initially emerged in the article demonstrating existence of solutions of differential equations .which appeared in the second quarter of the 18th century. Lateral on this technique was improved as a method of successive approximations which was extracted and abstracted as a fixed point theorem in the framework of complete normed space. It is stated that fixed point theory is initiated by Stefan Banach. Fixed point theorem have been used in many instances in approximation theory. Brosowski gave some application of fixed point theory for approximation.

Keywords: Fixed points; Approximation theory; Mappings; Normed linear space; Banach space; Sets; Common fixed points

1. Introduction

Let X denote the linear normed space with norm II.II and C be a subset of X. If $x_0 \in X$

be a point then y∈ C is called a best approximation to x_0 if

$$d(x_0,y)=d(x_0,C) \ldots\ldots(1)$$

$$\text{where } d(x_0,C)=\inf\{ \| x_0-y\| : y\in C \} \ldots\ldots(2)$$

In this case, the element y is called a best C-approximant to x_0. The set of all best

C-approximation to x_0 is denoted by $A_C(x_0)$. Thus, we have

$$A_C(x_0)= \{ y\in C \mid d(x_0,y)=d(x_0,C) \} \ldots\ldots(3)$$

A mapping f : X→X is a contraction if

$$\| fx - fy \| < \alpha \| x - y \| \ (\alpha < 1) \text{ ; and non-expansion if}\} \ldots\ldots(4)$$

$$\| fx - fy \| \ \| x - y \| \quad \forall x, y \in X \ldots\ldots(5)$$

Brosowski gave application of fixed point theory to approximation.

* Corresponding author: Sharad Pawar

1.1. Theorem

- Let f be a non-expensive linear operator on a normed linear space X.
- Let C be an f-invariant subset of X and x_0 an f-invariant point. If the set $A_C(x_0)$
- of best C-approximants to x_0 is non-empty, compact and convex then it contains an f-invariant point . Several authors improved the above theorem keeping in view of the following conditions.

a - The condition of f : X → Y …………………………..(6)

b - f : C → C

c - The condition on the set $A_C(x_0)$

Following Dhage and Shukla has proved a theorem.

Def. : A non – empty set X together with a function (7) ρ : X x X x X → (0 , ∞) is called a D-metric space with D- metric C , denoted by

(X , ρ) . If ρ satisfies

a - ρ(x ,y ,z) = 0 iff x=y=z (coincidence)

b - ρ(x ,y ,z) = ρ(ρ{ x ,y ,z}) ; where, ρ is a permutation of { x,y,z}

c ρ(x ,y ,z) ≤ ρ(x ,y ,a) + ρ(x ,a ,z) + ρ(a ,y ,z) ∀ x, y , z , a ∈ X

Some details along with some specific examples of D-metric space in Dhage .

A sequence { Xn } ⊂ X is called convergent and converges to a point x ∈ X if

lim ρ (x_m, x_n, x) = 0…………………..(9)

A complete D-metric space is one in which D-Cauchy sequence converges to a Point . It is known that D-metric ρ is continuous function on X^3 in the topology of D- metric converges which is Hausdorff . A mapping is called D-contraction if

a - ρ (fx ,fy ,fz) ≤ α ρ(x ,y ,z) (α < 1)…………….(10) b - ρ(fx ,fy ,fz) ≤ ρ(x ,y ,z) ∀ x ,y , z ∈ X

Dhage proved the following theorems.

Theorem

Let X be a D-metric space and f : X X a D – contraction .Thus f has a unique fixed Point . Let C be a subset of the normed linear space X and let If x_0 , y_0 ∈ X denoted by

D (x_0, y_0) = inf { C (x_0, y_0 , C) I C ∈ } …………………………(11)

An element z ∈ C is said to be a best approximation to x_0 and y_0 from C or closest to x_0 and y_0 from C if

ρ (x_0, y_0 , z) = D (x_0, y_0 , C)………………………..…………..(12)

Thus z is called best C- approximation to x_0 and y_0 and the set of all such best C – approximant to x_0 and y_0 is denoted by Ac(x_0, y_0) . Therefore ,we have

A_C(x_0, y_0) = { z ∈ C /ρ (x_0, y_0 , C) = D (x_0, y_0 , C) }………………………..(13)

Schandler has shown that

Theorem : - Let C be a non – empty , closed ,convex and bundled subset of a normed linear Space X and f: C → C be continuous and f (C) compact. Then f is a fixed point. Shukla proved the theorem as follows.

Theorem: Let X be a normed linear space C ⊂ X and { x_0, y_0 } ⊂ X . Let f : X → X .

a - f is non expansive …………… ..(14)

b - f: C → C

C - f { x_0, y_0 } → (x_0, y_0) is injective .

d - $A_C(x_0, y_0)$ is closed convex if $A_C(x_0, y_0)$ is compact. C contains an f invariant point closed to x_0 and y_0 . A they pove another theorem .

Theorem: Let X be Banach space { x_0, y_0 } ⊂ X , and C ⊂ X and let f : X → X assume that Banach space .

a - f is non – expansive and $A_C(x_0, y_0)$ ∪ { x_0, y_0 }……………………(15)

b - f { x_0, y_0 } → (x_0, y_0) is injective

f : C → C and set $A_C(x_0, y_0)$ is closed and star – shaped. f $A_C(x_0, y_0)$ is compact. Then f fixed point in C closest to x_0 *and* , y_0 from a closest element to the set { x_0, y_0 } from C .

Shukla has proved a theorem that

Let X be a normed linear space C ⊂ X and { x_0, y_0 } ⊂ X . let f : X → X mapping Satisfy the following condition .

a - f is non – expansive

b - f : C → C

c - f : { x_0, y_0 } → { x_0, y_0 } is injective ,

d - $A_C(x_0, y_0)$ is closed convex and f { $A_C(x_0, y_0)$ } is compact.

C contains an f – invariant closest to x_0, and y_0 . A set together with a function

If ρ is satisfies

a - ρ (x , y ,z) = 0 ⇔ x = y = z…………………(16)

b - ρ (x , y ,z) = ρ { x , y ,z } . ρ is a permutation of x , y and z .

c - ρ (x , y ,z) ≤ ρ (x , y ,a) + ρ (x , a ,z) + ρ (a , y ,z) ∀ x , y , z , a ϵ X

A sequence { x_n } ⊂ X is called convergent and converges to m ,n ρ (x_m , x_n , x) = 0

Again a sequence { x_n } ⊂ X is called D-Cauchy sequence converges to a point in it .

A mapping f : (X , ρ) →(X , ρ) is called D -contractive with respect to a mapping

g : (X , ρ) →(X , ρ) if ρ (fx ,fy ,fz) < ρ (gx ,gy ,gz) for all x ,y , z ∈ X for which

ρ (gx ,gy ,gz) ≠ 0 and non – expansive with respect to a g : (X , ρ)) → (X , ρ)

If ρ (fx ,fy ,fz) ≤ ρ (gx ,gy ,gz) for all x ,y ,z ∈ X .

Two maps f and g : (X , ρ) →(X , ρ) are called coincident if there is a sequence { x_n }

In X such that $\lim_n f x_n = \lim_n g x_n$ Similarly , they are called limit commutative or limit commuting .

If there is a sequence { x_n } ∈ X s.t. $\lim_n (fg)(x_n) = \lim_n (gf)(x_n)$

Again two maps f and g : (X , ρ) →(X , ρ) are called limit coincidentally commuting if their limit coincidence implies the limit commuting on X , i.e. there exists a sequence

$$\{ x_n \} \subset X \text{ such that } \lim_n f x_n = \lim_n g x_n$$

$$\lim_n (fg)(x_1) = \lim_n (gf)(x_1) \text{.............................} (17)$$

Similarly , two maps f and g : (X , ρ) →(X , ρ) are called coincidentally commuting if They cummute at coincidence points .

Finally , f : (X , ρ) →(X , ρ) is continuous iff for any sequence { x_n } in X , $x_n \to x$ Implies f $x_n \to$ fx Dhage has shown a theorem that if f , g be two continuous self maps of a compact D – metric space X satisfying further

a - f(X) ⊆ g(X) ;

b - { f , g ,} are coincidentally commuting ,

c - Then f and g have a unique common fixed point .

Let x_0 , y_0 ∈ X and

$$D (x_0 , y_0) = \inf \{ \rho (x_0 , y_0 , C) \in C \text{...................} (18)$$

An element z ∈ C is said to be a best approximant to x_0 and y_0 from C or closest to x_0 and y_0 from C if $\rho (x_0 , y_0 , C) = D (x_0 , y_0 , C)$

In case z is called a best C-approximant to x_0 and y_0 from C denoted by $A_C (x_0 , y_0)$ ....(20)

$A_C (x_0 , y_0) = \{ z \in C / \rho (x_0 , y_0 , z) = D ((x_0 , y_0 , C) \}$

Where we denote $A_C = (x_0 , y_0) = A_C (x_0 , y_0) \cup \{ x_0 , y_0 \}$

1.2. Main Result

(2.1) a) F is non-expansive

b) f : C → C

c) f { x_0 , y_0 } → (x_0 , y_0) is injective

d) $A_C (x_0 , y_0)$ is closed convex and f ($A_C (x_0 , y_0)$) is compact.

gf = fg and gx = u ϵ C and f(X) = v ϵ C

Let us define a D-metric ρ on the normed space as follows :

II fz – gx ll

(2.2) ρ (x, y, z) = α { II x – y II + II y – z II + II z – x II } ∀ x, y, z ϵ X

Since f is an non-expansive on X with respect to II . II , we have. .. (2.3) p(fx, fy, fz) = α { II fx – fy II + II fy – fz II + II fz – fx II ≤ α { II x – y II + II y – z II +II z – x II }

≤ ρ (x, y, z) ∀ x, y, z ϵ X

which shows that f is D-non expansive on X.

If f : $A_C(x_0, y_0) \rightarrow A_C(x_0, y_0)$ and x ϵ $A_C(x_0, y_0)$ be any point, then, we have

(2.4) ρ (fx ,fy ,fz) = ρ (fx ,fx_0 ,fy_0) ≤ ρ (x , x_0 , y_0) = D (x_0 , y_0)

And so fx ϵ $A_C(x_0, y_0)$

If f is non – expansive it is continuous on X .Now , if a continuous and f {$A_C(x_0, y_0)$} is compact in X . So the desired conclusion follows by an application of theorem (4) .

Theorem : - Let X be a Banach space , (x_0 , y_0) ⊂ X , C ⊂ X and let f : X → X .

Also , if we assume that

(2.5) a) f is non -expansive on $A_C(x_0, y_0) \cup \{x_0, y_0\}$

b) f :{ x_0 , y_0 } → (x_0 , y_0) is injective

c) f : C → C and

d) The set $A_C(x_0, y_0)$ is closed and star shaped and f {$A_C(x_0, y_0)$} is compact.

Then f has a fixed point in C closest to x_0 and y_0 .

Proof : - Define a D- metric C on the normed linear space by Eq . f defines such that

(2.6) f : $A_C(x_0, y_0) \rightarrow A_C(x_0, y_0)$, The theory

Let { t_n} be a sequence of real numbers such that 0 ≤ t_n < and for sufficiently

large n , t_n = , i.e. , $t_n \rightarrow 1$, n → ∞ .

{f_k}mapping$A_C(x_0, y_0)$ (2.7) f_k (x) = (1- t_k) q + f_k fx

For each K ϵ N from hypothesis (i) it follows that f and consequently f_k is Uniformly continuous on $A_C(x_0, y_0)$. Some $A_C(x_0, y_0)$ star-shaped w.r.t. the point

q ϵ $A_C(x_0, y_0)$, f_k (x) $A_C(x_0, y_0)$.

Consequently

f_k { $A_C(x_0, y_0)$ } ⊆ $A_C(x_0, y_0)$ for each K ϵ N .

Next , we show that { f_k , g } satisfy condition of (4.2.5) and its limit is coincidently commuting $A_C(x_0, y_0)$ for sufficiently large value of K. First weshall show that { f_k , ∞ } satisfy condition (3) on $A_C(x_0, y_0)$ for each K ϵ N . Since , f is non-expansive on $A_C(x_0, y_0)$ w.r.t. map , we have

(2.8) ρ ($f_k x, f_k y, f_k$z) = α[IIf_kx - f_ky II + II f_ky - f_kz II + II f_kz - f_kx II]

= α { t_k II $-f_z$ II + t_k II $f_z - f_x$ II }

≤ α t_k {II x – y II + II y – z II + II z – x II }

= α t_k { II gx – gy II + II gy – gz II + II gz – gx II }

= t_k ρ (gx , gy ,gz)

$\leq \rho$ (gx , gy ,gz) , $t_k < 1$

for all x , y , z ϵ A_C (x_0 , y_0) for which we find ρ (fx ,fy ,fz) ≠ 0 .

This shows that { f_k , ∞} satisfy the condition on A_C (x_0 , y_0) and hence are

D-contractive on A_C (x_0 , y_0) .Now , we show that { f_k , ∞} are limit coincidently

Commuting on A_C(x_0 , y_0) for sufficiently large value of K .

Assume that { f_k ,∞} are limit coincident on A_C(x_0, y_0) for large value of K .That is there is a sequence {x_n } in A_C(x_0, y_0) such that

(2.9) $\lim_n f_k x_n = \lim_n g\, x_n$ which yields

$\lim_n (\lim_k f_k x_n) = \lim_n f\, x_n = \lim_n g\, x_n$ Now ,

(2.10) $\lim_n f_k\, g x_n = \lim_k [(1 - t_k)q + t_k\, g\, x_n]$

$= \lim_k (1 - t_k)q + \lim_k t_k fg\, x_n$

Therefore by uniform continuity of g , we obtain

(2.11) $\lim_n (\lim_k t_k g\, x_n) = \lim_n f g x_n = \lim_n g f x_n = g(\lim_n f x_n)$

$= g(\lim_n (\lim_k f_k\, x_n)) = \lim_n (g \lim_k (f_k\, x_n))$

$= \lim_n (\lim_k g f_k\, x_n)$

$\lim_k \lim_n g f_k\, g x_n = \lim_n \lim g f_k\, x_n$

Which shows that f_k and g are coincident . Commuting on A_C(x_0, y_0) for Sufficiently large values n/v .

2. Conclusion

In this paper, we have discussed the application of fixed point theorems in approximation theory. It presents a few results in approximation theory using fixed point theorems. Fixed point theorems have been used in many instances in approximation theory.

Compliance with ethical standards

Acknowledgments

The author does thankful to the editor and anonymous reviewers for their valuable suggestions which improve the content of the paper.

References

[1] Brosowski , B . (1969) : Fixpunktstage in der approximation theory, Mathematica Cling 11 , 195 – 200 .

[2] Dhage A . (2001) : A result in best approximation theory , Indian J.Pure & Appl. Math.Soc.,32 ,1209 – 1216 .

[3] Dolbosco and Dominico (1999) : Some remarks of best approximation and fixed point ,Indian J.Pure & Appl.Math., 30,645 – 748 .

[4] Sharma , Sushil and Bhavna Deshpande (2002) : Approximation of fixed points and convergence of generalized iteration , Indian Math.Soc.,32 , 185 – 191.

[5] Shukla ,D.P. (2001) : Application of some fixed point theorems in approximation theory , Acta Ciencia Indica XXVII, M.,373-376.

[6] Singh S.P.(1979) : An application of fixed point theorem in approximation theory, 25, 89 -90