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An over view of geometric function theory (GFT) and analytic properties of meromorphic functions

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Abstract

This article presents an over view of Geometric Function Theory (GFT) and utilized the analytic properties of meromorphic functions. Geometric Function Theory is a branch of mathematics that focuses on the geometric interpretations and implications of analytic functions defined in the complex plane. Our exploration begins with an itemized discussion of key concepts within GFT, emphasizing their relevance and theoretical underpinnings. Central to our study is the investigation of meromorphic functions, which are functions that are analytic except for isolated singularities where they may have poles. We examined various classes of meromorphic functions and elucidate their properties, including their behavior near singularities and their broader geometric implications. A significant portion of our inquiry involves the Hadamard product of functions. This operation allows us to explore the combined effect of two analytic functions, considering their series expansions and how their product transforms under this operation. By studying the Hadamard transformation, we uncover analogues and interesting results that shed light on the interplay between analytic functions and their geometric representations. We also provide detailed diagrammatic descriptions of fundamental geometric shapes such as circles, open unit disks, and closed unit disks. These diagrams serve to visually illustrate key concepts and relationships within GFT, aiding in the understanding of how analytic functions behave in different spatial configurations. Our article offers a comprehensive exploration of Geometric Function Theory, emphasizing its foundational concepts and their applications in analyzing analytic and meromorphic functions.

Keywords: Geometric function theory; Analytic functions; Meromorphic functions; Hadamard product; Unit disk; Starlike functions; Convex functions

1. Introduction

Geometric function theory (GFT) is one of the most striking areas in mathematical analysis that has raised interest of many researchers since the beginning of 20th century. Teodor *et-tal* (2010),this branch of complex analysis is highly fascinating because of its applications in other fields, like model mathematical physics, fluid dynamics, fractional calculus, linear and nonlinear integreable system theory and theory of partial differential equations. Geometric Function Theory (GFT) deals with the theory of univalent functions and it is associated with geometry properties of analytic functions..

1.1. Applications of Geometric Function Theory

Conformal Mapping and Cartography: Conformal mappings, which preserve angles locally, are extensively used in cartography to create accurate map projections. By employing GFT principles, cartographers can transform the Earth's

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curved surface into a flat map while preserving shapes and angles, crucial for navigation and spatial analysis (Roshihan, et al 2009).

Complex Analysis in Mathematical Physics: GFT provides powerful tools for studying harmonic functions and potential theory, which are foundational in mathematical physics. These tools are essential in analyzing heat conduction, gravitational fields, and quantum mechanics, where understanding the behavior of analytic functions in complex domains is crucial (Cho and Srivastava, 2003).

1.2. Some basic definitions

1.2.1. Complex Plane

the set of all complex numbers is designed by $\mathbb{C} = \{z = x + iy : x, y \epsilon \mathbb{R}\}$. The field of complex number is the set $\mathbb{C}$equipped with the basic arithmetic operations of addition, subtraction, multiplication and division .

The imaginary part of a complex number z is written as $Im\ (z)$ hile its real part is denoted $Re\ (z)$.

1.2.2. Unit Disk

A disk is a region in a plane bounded by a circle. A unit disk is denoted $\Delta = \{z: |\boldsymbol{z}| < 1\}$i.e a set of all points z in the complex plane which lie in the $|\boldsymbol{z}| < 1$. A set of points inside a circle of radius r about $z = z_0$ is called an open disk or neighborhood of z_0 , open disk can be written as $|z - z_0| < r$. The set of all point in $|z - z_0| \leq r$ is the closed disk of radius r about $= z_0$. In other words, a disk is said to be closed when it includes the bounding circle and its said to be open when it does not.

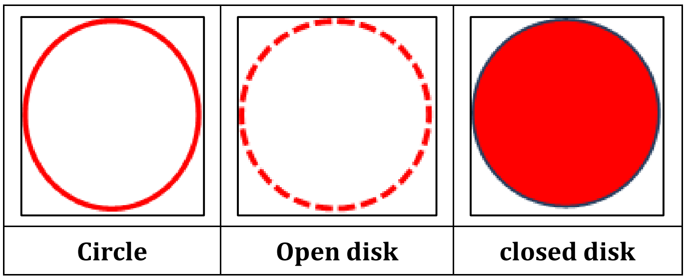

Figure 1 Circle, open and closed disk

1.2.3. Analytic function

Supposed that $\Delta = \{z: \boldsymbol{\epsilon}\, \mathbb{C} : |\boldsymbol{z}| < 1\}$denotes open unit disk in the complex plane $\mathbb{C}$. A function $f(z)$ of the complex variable is said to be analytic at a point z_0 if its derivative exist not only at z_0 but at every point at some neighborhood of z_0. A function $f(z)$ is analytic in a unit disk Δ if it is analytic every point in the complex plane $\mathbb{C}$, we say that $f(z)$ is entire function.An analytic function of complex variable z is also called monogenic, regular or holomorphic function.

Example 1.2.1

The function $\frac{1}{z}$ is analytic everywhere in the complex plane $\mathbb{C}$ except at $z = 0$ because its derivative does not exist at $z = 0$.

1.2.4. Differentiability of function of complex variables

A domain Ω is any connected open subset of the complex plane ($\mathbb{C}$) it includes the open upper half plane and the open disk. Supposed z_0 is an interior point on the domain of a function $f(z)$, the function $f(z)$ is said to be differentiable at point z_0 if the derivative $f'(z)$ exist at point z_0. In words, other the graph of $f(z)$does not have vertical tangent line at the point $\big(z_0, f(z_0)\big)$and it has no break point and angel but relatively smooth.

A function $f: \mathbb{C} \longrightarrow \mathbb{C}$ is said to be differentiable at $z = z_0$ if

$$f'\,(z_0) = \lim_{h \to 0} \frac{f(z_0 \ + h) - f(z_0)}{h} \ldots\ldots\ldots\ldots\ldots. \qquad (1.2.5)$$

Provided the limit exist and is independent of path as $h \longrightarrow 0$.

1.2.5. Continuity of function of complex variable

The property of continuity of function is exhibited by various aspect of nature such as the water flowing in the river at time t, the flow of blood and the growth of human being at time t, etc. A function $f(z)$ of complex variable z is said to be continuous at the point z_0 if for any given positive numberϵ there exist a number δ such that as $|f(z) - f(z_0)| < \epsilon$ whenever$|z - z_0| < \delta$. In other words, we say that a function $f(z)$ is continuous at a point z_0 when we can make the value of $f(z)$ become close to $f(z_0)$ by taking z close to z_0. It can be written $\lim_{z \longrightarrow z_0} f(z)= f(z_0)$.

A function $f(z)$ is said to be class C^k if the first K derivatives $f'(z), f''(z), f'''(z), ..., f(z)^k$ all exist and are all continuous. A function $f(z)$ is said to be of class C^∞ if the function is smooth and the derivative of $f(z)$ exists for all positive integers k.Most continuously differentiable functions are sometime said to be of classC^1. All the differentiable functions are continuous but not all continuous function are differentiable.

1.2.6. Univalent function S

Koebe (1907) initiated the theory of univalent functions. He found the range of all univalent functions S containing a common disk $|\boldsymbol{z}| < \frac{1}{4}$, the leading example of univalent function is the function $k(z)$ called the koebe function defined by $k(z)=\frac{z}{(1-z)^2} = z + 2z^2 + 3z^3+... = \sum_{n=1}^{\infty} \mathrm{n}z^{\mathrm{n}}$ ... ... (1.2.6)

It maps Δ onto the complex plane but does not contain the point - ∞ to - $\frac{1}{4}$ and so does not include any disk centered at 0 and radius greater than $\frac{1}{4}$. A function a $f(z)$ is said to be univalent in a domain Ω if it is one-to-one and analytic on Ω i that is given z and z' $f(z)=f(z') \Rightarrow z = z'$ and $f(z) \neq f \quad (z') \Rightarrow z \neq z'$.

1.2.7. Single and multi-valued functions

A function might be single valued or multi-valued depends on the domain of restriction. For example, $f(x)=(x + 1)^{\frac{1}{2}}$ has a single value for $x = 0, f(x) \epsilon \mathbb{R}$, but has the infinite solutions for $f(x) = e^{2\pi(1+n)i/2}$ $n \epsilon \mathbb{Z}$ in the complex filed. A function $f(z)$ of complex variable z is said to be single value function if $f(z)$ has only one value for each value of z in the domain of restriction while a function $f(z)$ is called a multi-valued function if f(z) has more than 0ne distinct values for each value of z in the domain of restriction some examples of multivalent functions include $\sqrt{z}, n\sqrt{z}$, $logz$, $\cos^{-1} z$ and $\sin^{-1} z$.

Normalized function.

If $f(z)$ is analytic at z=z_0 then the power series $\sum_{n=0}^{\infty} \mathrm{C_n}(\mathrm{z} - z_0)^{\mathrm{n}} = \mathrm{C_0} + \mathrm{C_1}(\mathrm{z} - z_0) + \mathrm{C_2}(\mathrm{z} - z_0)^2 + ...$ (1.2.7)

Equation (1.2.7) is called the Taylor series expansion of $f(z)$ aroundz_0. If $z_0 = 0$ we have Macaurin series expansion of $f(z)$ about z_0 = 0, where $c_n=\frac{f^n(z_0)}{n!}$.For z_0= 0 the Maclaurin series can be written as

$$f(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots = \sum_{n=0}^{\infty} c_n\ z^n. ,\text{.........................}(1.2.8)$$

From (1.2.8) $f(z) - c_0 = c_{1z} + c_2 z^2 + c_3 z^3 + ...$

Setting $\frac{f(z)-c_0}{c_1} = h(z)$and $\frac{c_n}{c_1} = {}_{b_n}\ bn, N = 2,3,4 ...$

We have

$$h(z) = z + \sum_{n=2}^{\infty} bnz^n \text{ ....}(\ 1.2.9)$$

Equation (1.2.9) is called a normalized function if $h(z)$ is one-to-one function and has the normalized form (1.2.9)

A smooth curve γ is a function $z: [a, b] \subset \mathbb{R} \longrightarrow \mathbb{C}$ such that $z(t) = x(t) + iy(t)$ where $x(t)$ and $y(t)$ are real and hence differentiable. It is simple (non- self-intersecting if $z(t_1) \neq z(t_2)$. for a≤ $t_1 < t_2 < b$and it is closed if $z(a) = z(b)$. Any smooth curve is directed or oriented with initial point $z(a)$ and endpoint $z(b)$.

1.2.8. Hadamard Product of Analytic function

Given two functions $\phi_1(z)$ and $\phi_2(z)$ which belong to the class of $\mathcal{A}$ defined by $\phi_1(z) = \mathrm{z} + \sum_{k=2}^{\infty} \mathrm{b_k}\, \mathrm{z^k}$, $\phi_2(z) = \mathrm{z} + \sum_{k=2}^{\infty} \mathrm{c_k}\, \mathrm{z^k}$ then the Hadamard product or (convolution) of functions $\phi_1(z)$ and $\phi_2(z)$ denoted $(\phi_1(z) * \phi_2(z)) = \mathrm{z} + \sum_{k=2}^{\infty} \mathrm{b_k c_k}\, \mathrm{z^k} z\epsilon\Delta$ 1.2.9

Example 1.2.3

The Hadamard product of the horizontal strip map $\phi_1(z) = \frac{1}{2} log\left(\frac{1+z}{1-z}\right)$ and the function $\phi_2(z) = \frac{z(1+z^2)}{(1-z^2)^2}$ can be obtained as follows

From Maclaurin series

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + z^4 + \ \dots \ z\epsilon\Delta$$

Integrating both sides we have

$$\log(1-z) = -\sum_{k=0}^{\infty} \frac{\mathrm{z^{k+1}}}{\mathrm{k+1}}$$

similarly$\frac{1}{1+z} = 1 - z + z^2 - z^3 + z^4 + \cdots \ z\epsilon\Delta$

By integrating we have

$$\log(1+z) = \sum_{k=0}^{\infty} (-1)^{\mathrm{k}} \frac{\mathrm{z^{k+1}}}{\mathrm{k+1}}$$

$$log\left(\frac{1+z}{1-z}\right) = \log(1+z) - \log(1-z) = \sum_{k=0}^{\infty} (-1)^{\mathrm{k}} \frac{\mathrm{z^{k+1}}}{\mathrm{k+1}} + \sum_{k=0}^{\infty} \frac{\mathrm{z^{k+1}}}{\mathrm{k+1}}$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} + \cdots + z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \frac{z^5}{5} + \frac{z^6}{6} + \cdots$$

$$= 2z + \frac{2z^3}{3} + \frac{2z^5}{5} + \frac{2z^7}{7} += 2\sum_{k=0}^{\infty} \frac{\mathrm{z^{2k+1}}}{\mathrm{2k+1}}$$

$$\frac{1}{2} log\left(\frac{1+z}{1-z}\right) = \sum_{k=0}^{\infty} \frac{\mathrm{z^{2k+1}}}{\mathrm{2k+1}}$$

Now consider $\frac{1}{1-z^2} = \ 1 + z^2 + z^4 + z^6 + \cdots$

Multiply both sides by z, we have

$$\frac{z}{1-z^2} = \ z + z^3 + z^5 + z^7 + \cdots$$

Differentiating both sides yields

$$\frac{1+\ z^2}{(1-z^2)^2} = \ 1 + 3z^2 + 5z^4 + 7z^6 + \cdots$$

Multiply both sides by z

$$\frac{z(1+\ z^2)}{(1-z^2)^2} = \ z + 3z^3 + 5z^5 + 7z^7 + \cdots$$

$$\frac{z(1+\ z^2)}{(1-z^2)^2} = \sum_{k=0}^{\infty}(2k+1)z^{2k+1}$$

Finally, the Hadamard product of $\phi_1(z)$ and $\phi_2(z)$ is

$$(\phi_1(z) * \phi_2(z)) = \frac{1}{2}log\left(\frac{1+z}{1-z}\right) * \left(\frac{z(1+\ z^2)}{(1-z^2)^2}\right) = \sum_{k=0}^{\infty}\frac{\mathrm{z}^{2\mathrm{k}+1}}{2\mathrm{k}+1} * \sum_{k=0}^{\infty}(2k+1)z^{2k+1}$$

$$= \left(z + \frac{z^3}{3} + \frac{z^5}{5} + \frac{z^7}{7} + \cdots\right) * (z + 3z^3 + 5z^5 + 7z^7 + \cdots) = z + z^3 + z^5 + \cdots z(1 + z^2 + z^4 + \cdots) = \frac{z}{1-z^2}$$

Definition Miller and Mocanu (2000)

Let ψ : $\mathbb{C}^3 \times \Delta \to \mathbb{C}$ and g(z) be univalent in

Δ. If p(z) is analytic in Δ and satisfies the second order differential equation subordination

$$\psi(p(z), zp'(z) + z^2p''(z) : z) \qquad (2.1.10)$$

Then $p(z)$ is called a solution of the differential subordination (2.1. 10). An analytic function $q(z)$ is called a dominant of the solution of the differential subordination (2.1.10) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (2.1.10). A univalent dominant $\bar{q}(z)$ that satisfies$\bar{q}(z) \prec q(z)$ is said to be the best dominant of (2.1.10) for all $q(z)$ in (2.1.10)

Definition let ψ: $\mathbb{C}^4 \times \Delta \to \mathbb{C}$ and suppose that the function $g(z)$ is univalent in Δ and the univalent function $p(z)$ satisfies the third order differential subordination

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z) : z) \prec g(z) \ldots\ldots\ldots\ldots\ldots (2.1.11)$$

Then $p(z)$ is called a solution of the differential subordination (2.1.11), a given univalent function $q(z)$ is called a dominant of the solution of (3.1.5) if $p(z) \prec q(z)$ for every $p(z)$ satisfying condition (2.1.11).A dominant $\tilde{q}(z)$ that satisfies the condition $\tilde{q}(z) \prec q(z)$ for all dominant $q(z)$ of (2.1.11) is called the best dominant

Definition Miller and Mocanu (2000) let Ω be a set in $\mathbb{C}$, $q\epsilon Q$ and $n\epsilon\mathbb{N}$ / {2}, the class of admissible function $\psi_n[\Omega, \mathrm{q}]$ consists of those functions ψ: $\mathbb{C}^4 \times \Delta \to \mathbb{C}$, which satisfy the following admissible conditions

$$\psi(r, s, t, u : z) \notin \Omega$$

whenever

$$r = q(z), s = k\zeta q'(\zeta), \mathrm{Re}\left(\frac{t}{s} + 1\right) \geq kRe\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right)$$

$$Re\left(\frac{u}{s}\right) \geq k^2 Re\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right)$$

for $z\epsilon\Delta, \zeta\epsilon\partial\Delta/\ E(q)$ and $k \geq n$

Definition let Miller and Mocanu (2003) Ω be a set in $\mathbb{C}$, also let $q\epsilon\mathcal{H}[a,n]$ and $q'(z) \neq 0$, the class $\psi'_n[\Omega, q]$ of admissible functions consists of those functions $\psi: \mathbb{C}^4 \times \Delta \rightarrow \mathbb{C}$ that satisfy the following admissible conditions

$$\psi(r, s, t, u: z)\epsilon\Omega$$

Whenever

$r = q(z)\,, s = \frac{zq'(z),}{m} \operatorname{Re}\left(\frac{t}{s} + 1\right) \leq \frac{1}{m} Re\left(\frac{zq''(z)}{q'(z)} + 1\right)$

$$Re\left(\frac{u}{s}\right) \leq \frac{1}{m^2} Re\left(\frac{z^2 q'''(z)}{q'(z)}\right)$$

Whenever $z\epsilon\Delta, \zeta\epsilon\partial\Delta$ and $m \geq n \geq 2$

Definition . Salagean 1981 introduced the following operators differential operators

$$D^0 f(z) = f(z),$$

$$f(z)D^1 f(z) = zf'(z), \ldots,$$

$$D^n f(z) = D\left(D^{n-1} f(z)\right)$$

$(n \in N\ 1,2,3, \ldots)$ Let Δ

1.3. Certain classes of analytic function

Several authors considered classes defined by geometric conditions since the Bieber Beach conjecture was so complex to settle. Highly relevant among them are the classes of starlike convex functions, close to convex functions, class of close - to – convex functions, clase of α - starlike and α – convex functions and class Quasi-convex functions.

1.3.1. The Class of Starlike functions $\boldsymbol{S}^*$

A given domain Ω in the complex plane which includes the origin in called starshaped with respect to the oringin if there exist a point $z_0\epsilon\ \Omega$ the point $\lambda z_0\epsilon\ \Omega$ for all real λ statisfying $0 \leq \lambda \leq 1$ in other words, if Ω contains z_0 then it eventually contains the line segment joining z_0 to the origin.

Therefore, a starlike function $f(z)$ is a function which is analytic and injective Δ which maps Δ conformally onto a star shaped domain so that $f(0)$= 0 and $f'(0) \neq 0$.

Theorem 1.3.1 (Duren 1983) A function $f(z)$ is starlike function if and only

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0\ for\ |z| < 1 \text{ ……………………} \qquad (1.3.1)$$

A set H in the complain plane $\mathbb{C}$ is said to be starlike with respect to a point $z_0 \in H$ if the linear segment joining z_0 to every point $z \in H$ lies entirely in H i.e

$$(1 - \lambda)z + \lambda z_0 \in H, 0 \leq \lambda \leq 1$$

1.3.2. The Class of Convex functions $\boldsymbol{C}^*$

Convex functions play an important role in many areas of mathematics such as optimization problem for determine minimum of a function and also in probability theory and calculus of variation. A set $H \in \mathbb{C}$ is said to be convex if $H \in S^*$ and the linear segmanet joining any two point of H lies entirely in H, i.e

$$(1 - \lambda)z_1 + \lambda z_2 \in H \forall z_1, z_2 \in H\ 0 \leq \lambda \leq 1$$

Definition 1.3.1 (Duren 1983)

Let $f(z)$ be a starlike function then $f(z)$ maps a unit disk (Δ) onto a convex domain, if and if and only if

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > 0\ for\ |z| < 1. \qquad (1.3.2)$$

A function $f\,;\ (A,B) \longrightarrow \mathbb{C}$ is convex if the function $\psi(z_1,z_2) = \frac{f(z_1)-f(z_2)}{(z_{1-z_2})} = \psi(z_1,z_2)$is monotonically increasing that is for $a < s < t < u < b$

$$\frac{f(t)-f(s)}{(t-s)} \le \frac{f(u)-f(s)}{(u-s)} \le \frac{f(u)-f(t)}{(u-t)}$$

The analytic connection between the convex function and the starlike was discovered by Alexander in 1915.

1.3.3. The Class of Close - to – Convex function

The class of close to convex function was discovered by W. Kaplan 1952. Its an interesting class of univalent function which contains starlike functions with a simple geometric description.

Definition 1.3.3 A function $f(z)$ which is analytic in an open unit disk is said to be close-to- convex function if there exist a convex function $h(z)$ such that

$$Re\left\{\frac{f'(z)}{h'(z)}\right\} > 0\ for\ |z| < 1. \qquad (1.3.3)$$

In this work we denote the class of close – to- convex functions by C^{**} where $f(z)$ is normalized by usual conditions $f(0)\ =\ 0, f(0) \neq 0.$ its imperative to note that every starlike function is close to convex function and every convex function is obviously close to convex. Every starlike functions satisfies the condition $f(z) = h'(z)z$ and

$$Re\left\{\frac{f'(z)}{h'(z)}\right\} = \ Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0\ for\ |z| < 1\ ....(\ 1.3.4)$$

Geometrically, close-to- convex functions are functions whose tangent to any given curve $T(r) = \left\{f\left(re^{i\theta}\right): 0 \le \theta \le 2\pi\right\}\ for$ each

$r < 1$ and never turn back onto itself as much as π radian.

Theorem 1.3.2 (Derek 1967) Every close-to convex function is univalent.

Proof suppose $\varphi(w)$ is the inverse of a convex function $y(x)$, and let $X(w) = f(\varphi(w))$, where $X\ (w)$ is analytic is some convex domain

$$X'(w) = f'(\varphi(w))\varphi'(w) = \frac{f'(z)}{y'(z)} \text{from 1.3.3 } ReX'(w) > 0$$

Suppose $w_1, w_2 \in \mathbb{R}$ then $w_2 \neq w_1 \Longrightarrow X\ (w_2) \neq X\ (w_1)$

So that $Re\left\{\frac{X\ (w_2)-X\ (w_1)}{w_2-w_1}\right\} = \int_0^1 ReX'(w_1 + t(w_2 - w_1))dt > 0$

Hence $f(z) = X(y(w))$ is univalent in a unit disk Δ

1.3.4. The Clase of $\boldsymbol{\alpha}$ *- Starlike and* $\boldsymbol{\alpha}$ *– Convex functions*

Robertson introduced the classes of starlike and convex function of order α denoted by $s^*\ (\alpha)$ and $\boldsymbol{C}^*(\ \alpha\ )$ where $0 \le \alpha \le 1.$

$$s^*\ (\alpha) = \ \left\{f \in \mathcal{A}: Re\left(\frac{zf'(z)}{f(z)}\right)\right\} > \alpha\ 0 \le \alpha \le 1.\ z\boldsymbol{\epsilon}\Delta$$

$$\boldsymbol{C}^*(\alpha) = \left\{ f \in \mathcal{A}: Re\left(\frac{zf'(z)}{f'(z)}\right)\right\} > \alpha\ 0 \leq \alpha \leq 1.\ z\boldsymbol{\epsilon}\Delta$$

The starlike and convex function of order α satisfy the following conditions $S^*(0)= s^*$ and$\boldsymbol{C}^*(0) = \boldsymbol{C}^*$ where s^* an d $\boldsymbol{C}^*$ have their unsual meanings.

1.3.5. The Class Quasi-Convex functions.

In 1980 Nooret-al introduced the class of Quasi-convex functions denoted by k defined by $Re\left(\frac{(zf'(z))'}{g'(z)}\right) > 0$ for $g(z)\boldsymbol{\epsilon C}^*$ ***and*** $z\epsilon\Delta$.

1.4. Subordination

let the functions $h_1(z)$ and $h_2(z)$ be regular in a unit disk Δ and $h_2(z)$ is univalent in $|z| < 1$. Let Ω_1 and Ω_2 denote the domains in the $w-$plane onto which the unit disk $|z| < 1$ is mapped by $w = h_1(z)$ and $w = h_2(z)$ respectively. If $h_1(0) = h_2(0)$ and Ω_1 is contained in Ω_2 we say that $h_1(z)$ is subordinate to $h_2(z)$. Symbolically we write, $h_1(z) \prec h_2(z)$.

Miller and Macanu (2000) let f and F be members of $\boldsymbol{\mathcal{H}}$. The function f is subordinate to F, write $f \prec F$ or $f(z) \prec F(z)$ if there exists a function w analytic inΔ with $w(0) = 0$ and $|w(z)| < 1$ and such that $f(z) = F(w(z))$. if F is univalent, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(\Delta) \subset F(\Delta)$.

1.4.1. The General hypergeometric function

Given two analytic functions $g(z) = z^p + \sum_{k=p+1}^{\infty} \mathrm{b_k}\,\mathrm{z^k}$ and

$h(z) = z^p + \sum_{k=p+1}^{\infty} \mathrm{c_k}\,\mathrm{z^k}$ for $z \in \Delta$ and $p \in \mathbb{N}$ the Hadamard product (or convolution) of g and h is defined by

$(g * h)(h) = z^p + \sum_{k=p+1}^{\infty} \mathrm{b_k c_k}\,\mathrm{z^k}$ for $\alpha_i \in \mathbb{C}$ where $(i = 1,2,3 \dots, q)$ and $\beta_j\epsilon\mathbb{C}$ where $(j = 1,2,3 \dots, r)$, $\{\beta \neq 0, -1, -2, \dots\}$ the generalized hypergeometric function

${}_q^rF(\alpha_1, \alpha_2, \dots, \alpha_q: \beta_1, \beta_2, \dots, \beta_r: z)$ is defined by infinite series

$${}_q^rF(\alpha_1, \alpha_2, \dots, \alpha_q: \beta_1, \beta_2, \dots, \beta_r: z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_\mathrm{n}, \dots, (\alpha_q)_\mathrm{n} \mathrm{z^n}}{(\beta_1)_\mathrm{n}, \dots, (\beta_q)_\mathrm{n} \mathrm{n!}} \dots\dots\dots\dots\dots\dots \qquad (2.1.1)$$

$(q \leq r+1, q, r \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}: z \in \Delta)$, where $(e)_n$ is the Pochamer symbol or (shifted factorial which is expressed in terms of Gamma function by

$$(e)_n = \frac{\Gamma(e+n)}{\Gamma(e)} = \begin{cases} 1\ if\ n = 0 \\ e(e+1) \dots (e+n-1)\ if\ n \in \mathbb{N}, e \in \mathbb{C} \end{cases} \qquad (2.1.2)$$

1.4.2. Integral Operators

Let $f\epsilon\mathcal{H}[0, \Omega]$, that is f is analytic in Δ with $f(z) = a_n z^n + a_{n+1} z^{n+1} + \cdots$

The Alexander operator[43] is defined by

$$A[F](z) = \int_0^z \frac{f(t)dt}{t}. \qquad (2.1.3)$$

The Libera operator is defined by

$$L[F](z) = \frac{2}{z} \int_0^z f(t)dt. \qquad (2.1.4)$$

Bernardi operator which generalizes the Libera operator is defined by

$$L_r[F](z) = \frac{r+1}{r} \int_0^z f(t) t^{r-1} dt 2.1.5$$

$r = 1,2,3 \ldots$ where each of these operators is well defined on $\mathcal{H}[0,\Omega]$ and maps f $into$ $\mathcal{H}[0,\Omega]$

1.4.3. Functions with positive real part

Let b denotes the class of functions

$P(z) = 1 + b_1 z + b_1 z^2 + \cdots$, which are regular in Δ, then $P(z)$ has positive real part is $Re\{P(z)\} > 0$ in Δ.

THEOREM 2.1.1

let the functions $h_1(z)$ and $h_2(z)$ be defined as given in (2.1.0) and

$h_1(z) \prec h_2(z)$ in Δ, then for all $q > 0$ and $0 < r < 1$

$$\int_0^{2\pi} \left|h_1(re^{i\theta})\right|^q \leq \int_0^{2\pi} \left|h_2(re^{i\theta})\right|^q. \qquad (2.1.4)$$

THEOREM 2.1.2

If $f \in \mathcal{A}$, then the following sharp implications hold

$$f'(z) \prec p'(z)$$

$$\Uparrow\Downarrow$$

$$\frac{zf''(z)}{f'(z)} \prec \frac{zp''(z)}{p'(z)} \frac{f(z)}{z} \prec \frac{p(z)}{p}$$

$$\Downarrow\Uparrow$$

$$\frac{zf'(z)}{f(z)} \prec \frac{zp'(z)}{p(z)}$$

2. Certain properties of meromorphic function

A meromorphic function is a function for each point in the domain has a unique value in the range (single- valued function) that is analytic in all but possibly a discrete subset of its domain and at some singular points whose values must tend to infinity.

For integer $p \geq 0$, denote by $\Sigma_p$the class of meromorphic functions defined in $\Delta \equiv \Delta\backslash\{0\}$, which are of the form

$$f(z) = \frac{1}{z} + a_p z^p + a_{p+1} z^{p+1} + \cdots \ldots \ldots \ldots \ldots \qquad (3.1.1)$$

And let $\Sigma = \Sigma_0$. A function $f \epsilon \Sigma$ is said to be starlike if it is univalent and the compliment of $f(\Delta)$ is starlike with respect to the origin. Denote by Σ^* the class of such functions. If $f \epsilon \Sigma$, then it is well known that $f \epsilon \Sigma^*$ if and only if

$\mathrm{Re}\left\{\frac{-zf'(z)}{f(z)}\right\} > 0$, for $z \epsilon \Delta$

We note that $f \epsilon \Sigma^*$ implies $f(z) \neq 0$ for $z \epsilon \Delta$.

For $\beta < 1$, let

$$\textstyle\sum_p^*(\beta) = \left\{f(-\sum p): Re\left[\frac{-zf'(z)}{f(z)}\right] > \beta, z\epsilon\Delta\right\} \ldots\ldots\ldots\ldots\ldots (3.1.2)$$

In addition, let $\Sigma^*(\beta) = \sum_0^*(\beta)$ and $\sum_p^*(0)$. Note that for $\beta < 0$ functions in this class need not be univalent in Δ.

Firstly, we will use the theory of differential subordinations to obtain relevant conditions for Σ_p to be starlike functions.In the second segment we will use the same theory to study the results of certain integral operators on subclass of Σ_p

2.1. Lemma 3.1.1

Let n be a positive integer and β be real, with $0\leq \beta < n$. let q be analytic in Δ, with $q(0) = 0$ and $q'(0) \neq 0$ and

$$Re\,\frac{zq''(z)}{q'(z)} + 1 > \frac{\beta}{n} \text{ ………………… (3.1.3)}$$

Define the function $h(z)$ as

$$h(z) = nzq'(z) - \beta q(z) \text{ …………………… (3.1.4)}$$

if $p\epsilon[0, n]$ and

$$zp'(z) - \beta p(z) \prec nzq'(z) - \beta q(z) = h(z). \text{ ……………… (3.1.5)}$$

Then, $p(z) \prec\ q(z)$ and this result is sharp.

Proof

Condition (3.1.3) implies that q is convex. From (3.1.3) and (3.1.4) we obtain

Re$\frac{h'(z)}{q'(z)} = n\left[\frac{zq''(z)}{q'(z)} + 1\right] - \beta$, which shows that $h(z)$ is close-to-convex univalent function.

We will use a subordination chain type argument to prove this lemma and without loss of generality, we can assume that $q(z)$ satisfies the conditions of the lemma on a closed disk Δ.

$$\text{The function } L(z,t) = (n+t)zq'(z) - \beta q(z) \text{ …………………(3.1.6)}$$

Is analytic in Δ for all $t \geq 0$, and is continuously differentiable on $[0,\infty)$ for all

$z\epsilon\Delta$. A simple calculation yields $a_1(z) = \frac{\partial L}{\partial z}(0,t) = (n+t-\beta)q'(0)$, hence $a_1(z) \neq 0$ and $\lim_{t\to\infty}|a_1(z)| = \infty$. From (3.1.4) we obtain

$$Re\left[\frac{z(\partial L\backslash\partial z)}{\partial L\backslash\partial L}\right] = (n+t)Re\left[\frac{zq''(z)}{q'(z)} + 1\right] - \beta > 0\,;\ z\epsilon\Delta \text{ and } t \geq 0. \text{ ………………… (3.1.7)}$$

$L(z,t)$ is a subordination chain and so we have $L(z,t_1) \prec L(z,t_2)$, where $0\leq t_1 0\leq t_2$. From(3.1.4) and (3.1.5) we have $L(z,0) = h(z)$. Hence, $L(\zeta,t) \notin h(\Delta)$, for$|\zeta| = 1$, and $t \geq 0$.

Now assume that p is not subordinate to q and there exist a point $z_0\epsilon\Delta$, $\zeta_0\epsilon\partial\Delta$, and and $m \geq 1$

such that

$$z_0p'(z_0) - \beta p(z_0) = m\zeta_0 q'(\zeta_0) - \beta q(\zeta_0) = L(\zeta_0, m-n) \notin h(\Delta).$$

Since this contradicts (3.1.5) we have $p(z) \prec\ q(z)$. The function $p(z) = q(z^n)$.

Shows that this subordination is sharp.

2.2. THEOREM 3.1.1

let n be a positive integer and let q be analytic function in Δ, with $q(0) = 0$ and $q'(0) \neq 0$ and

$$\frac{zq''(z)}{q'(z)} + 1 > \frac{1}{n+1},$$

Define the function $h(z)$as

$$h(z) \equiv (n+1)zq'(z) - q(z), \text{……………………..(3.1.7)}$$

if $f \epsilon \Sigma_p$, then

$z^2 f'(z) + 1 \prec h(z) \Rightarrow zf(z) - 1 \prec q(z)$ and this result is sharp.

Proof

Let $p(z) = zf(z) - 1$

Then $p\epsilon$[0,n+1] and $z^2 f'(z) + 1 = zp'(z) - p(z)$

The conclusion of the theorem follows by applying lemma 5.1.1 with $\beta = 1$ and replacing n by $n+1$

Corollary 3.1.1 If $f\epsilon\ \Sigma_p$ with $p \geq 1$, and if $M > 0$ then $|z^2 f'(z) + 1| < M \Rightarrow\ |zf(z) - 1| < \frac{M}{n}$ and this result is sharp.

Proof

If we take $q(z) = \frac{Mz}{n}$, then (3.1.7) becomes $h(z) = Mz$ and the result follows from theorem 3.1.1 .

2.3. Starlikeness of a meromorphic function

Al-Amiri and Mocanu, (1995) proved that if $f\epsilon\Sigma_p$ and $|z^2 f'(z) + 1| < 1$ then f is univalent in Δ. By applying our previous results we can obtain a similar simple criterion for starlikeness of a meromorphic function.

2.3.1. THEOREM 3.2.1

1 If $f\epsilon\Sigma_p$ with $n \geq 1$, then

$$|z^2 f'(z) + 1| < \frac{n}{\sqrt{n^2+1}} \Rightarrow f\epsilon{\Sigma_p}^*$$

Proof

Let 0< $M \leq \frac{n}{\sqrt{n^2+1}}$, suppose $f\ \epsilon\ \Sigma_p$ satisfies

$$|z^2 f'(z) + 1| < M. \text{ ………………….. (3.2.2)}$$

If $p(z) = zf(z)$ then by corollary 3.1.1

$$|p(z) - 1| < \frac{M}{n} < 1. \text{ …………………..(3.2.3)}$$

Which implies $p(z) \neq 0$. Let $p(z) = \frac{-zf'(z)}{f(z)}$ then, $p\ \epsilon$ [0,n+1] and (3.2.2) can be written in form

$$|P(z).p(z) - 1| < M. \text{ ………………….. (3.2.4)}$$

We claim that this inequality implies Re $p(z) > 0$, for all $z\ \epsilon\ \Delta$.

If this is false, then there exists a point $z_0\ \epsilon\Delta$ such that $p(z_0) = i\rho$, we will show that at such a point the negation of condition (3.2.4) holds

$$|P(z)i\rho - 1| \geq M. \text{ ……………….. (3.2.5)}$$

That is for real ρ. This inequality is equivalent to $|p(z_0)|^2\rho^2 + 2(Im)p(z_0)\rho + 1 - M^2 \geq 0$. And this holds for all real ρ if and only if

$$(Im(z_0))^2 \leq (1\text{-}M^2)|p(z_0)|^2 \text{ . ...(3.2.6)}$$

A simple geometric argument shows that the inequality (3.2.4) implies that

$$(Im(z_0))^2 \leq \frac{M^2}{n^2}|p(z_0)|^2.$$

Because of the definition of M given in (3.2.1) this forces inequalities (3.2.5) and (3.2.6) to hold. Thus we have a contradiction of (3.2.4) therefore Re $p(z) > 0$ and $f\epsilon\Sigma^*$.

Example 1

Let $f(z) = \frac{1}{z}+\lambda(z - sinz)$ in this case $f\epsilon\ \Sigma_3$ and

$$|z^2f'(z) + 1| = |\lambda||1 - cosz| = 2|\lambda|\left|sinh^2\left(\frac{z}{2}\right)\right| < 2|\lambda|\left|sinh^2\left(\frac{1}{2}\right)\right|$$

. Hence by theorem 3.2.1, if

$$|\lambda| = \frac{6e}{(e-1)^2\sqrt{10}} = 1.746 \ldots,$$

Then $\frac{1}{z}+\ \lambda(z - sinz)\epsilon\Sigma^*{}_3$

2.3.2. Lemma 3.2.1

let n be a positive integer, $n \geq 3$,and let $p\epsilon\mathcal{H}[1, n]$ satisfy

$p(z) \prec R_n(z) \equiv \frac{1+z}{1-z} - \frac{2nz}{1-z^2}$ and satisfies the differential equation

$$-zp'(z) + \ P(z).p(z) = 1$$

Then Re $p(z) > 0$.

From its definition we see that R_n is equivalent in Δ, $R_n(0) = 1$ and $R_n(\Delta)$ is the complex plane with slits along the half-lines Re$\omega = 0$ and $|Im\{\omega\}| \geq \sqrt{n(n-2)}$.

2.3.3. THEOREM 3.2.2

Let $f\epsilon\Sigma_p$, if $n \geq 2$ and $zf(z) \neq 0$.

if $-\left[\frac{zf''(z)}{f'(z)} + 1\right] \prec \frac{1+z}{1-z} - \frac{2(n+1)z}{1-z^2}$ then $f\epsilon\ \Sigma_p^*$.

Proof

If we let $p(z) = \frac{-f(z)}{[zf'(z)]}$, then $p(z) \neq 0, p\epsilon$[0,n+1] and we have

$\frac{zp'(z)}{p(z)} = -\frac{1}{p(z)} - \left[\frac{zf''(z)}{f'(z)} + 1\right]$ if we set $p(z) = -\left[\frac{zf''(z)}{f'(z)} + 1\right]$, then we deduce that $p\epsilon\ \mathcal{H}$[0,n+1] and $-zp'(z) + P(z).p(z) = 1$

Applying (3.2.1) we obtain

Re $p(z) > 0$ and Re$\left[\frac{1}{p(z)}\right] > 0$, and hence $f\epsilon\Sigma_p^*$

2.4. Some analogous results for meromorphic functions

Let us consider the operator defined by

$$I_{\gamma}[f](z) \equiv \frac{\gamma}{z^{\gamma+1}}\int_0^z f(t)t^{\gamma}dt = \gamma\int_0^1 f(u,z)u^{\gamma}du \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (3.3.1)$$

With $\gamma \in \mathbb{C}$ and Re $\gamma \geq 0$. It is easy to see that $I_{\gamma}: \Sigma \to \Sigma$.

2.4.1. THEOREM 3.3.1

let $0 \leq \alpha < 1$ and $0 < \gamma \leq 1$ if $f \epsilon \Sigma^*(\alpha)$ and $I_{\gamma}[f](z) \epsilon \Sigma^*(\beta)$ where

$$\beta = \frac{1}{4}\left[2\alpha + 2\gamma + 3 - \sqrt{[2(\gamma-\alpha)+1]^2 + 8\gamma}\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.3.2)$$

Proof

The conditions $\alpha < 1$ and $0 < \gamma$ are needed to imply that $\beta < 1$. let $f\epsilon\Sigma^*(\alpha)$ and let $F = I_{\gamma}[f]$ where I_{γ} is defined by (3.3.1). We shall first prove that $F(z) \neq 0$ for $z\epsilon\Delta$. this will eliminate the difficulty referred to above since $f\epsilon\Sigma^*(\alpha)$, we have $f(z) \neq 0$ for $z\epsilon\Delta$ and a simple computation shows that $g = \frac{1}{f}\epsilon S^*(\alpha)$ for $\alpha < 1$. If we define $h(z) = z\left[\frac{g(z)}{z}\right]^{\frac{1}{(1-\alpha)}}$. Then, $h(z)\epsilon S^*$. Apply Goluzin's subordination result we obtain $\left[\frac{h(z)}{z}\right]^{\frac{1}{2}} \prec \frac{1}{1+z}$.

From the relation between h, g *and* f we obtain $\frac{h(z)}{z} \prec (1+z)^{2(\alpha-1)}$ and $zf(z) \prec (1+z)^{2(\alpha-1)}$ and since $0 \leq \alpha < 1$, we have $zf(z) \prec (1+z)^2$ combining this with $\underset{|z|=1}{\text{Min}}\, Re(1+z)^2 = -\frac{1}{2}$ We deduce that $\text{Re}[zf(z)] > -\frac{1}{2}$. Differentiating (3.3.1) we obtain

$$(\gamma+1)F(z) + zF'(z) = \gamma f(z). \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.3.5)$$

Let $p(z) = zF(z)$, then $p\epsilon[1,1]$ and (3.3.5) becomes $\gamma p(z) + zp'(z) = \gamma zf(z)$, hence, we have

$$\text{Re}\psi\big(p(z), zp'(z)\big) = Re\left[p(z) + \frac{zp'(z)}{\gamma} + \frac{1}{2}\right] > 0. \ldots\ldots\ldots\ldots\ldots\ldots (3.3.6)$$

Where $\psi(r,s) = r + \frac{s}{r} + \frac{1}{2}$.

To show that Re$p(z) > 0$ it follows immediately since $0 < r < 1$,

Implies that Re$\leq \psi(i\rho,\sigma) = Re\left[i\rho + \frac{\sigma}{\gamma} + \frac{1}{2}\right]\frac{1+\rho^2}{2\gamma} + \frac{1}{2} \leq 0, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.3.7)$

When $\sigma \leq -\frac{(1+p^2)}{2}$, for $p\epsilon\mathbb{R}$. Hence from (3.3.6) we deduce that Re$p(z) > 0$, which implies $F(z) \neq 0$ for $z\epsilon\Delta$ we next determine β such that $F \in \Sigma^*(\beta)$. If we define p by

$$\frac{-zF'(Z)}{F(z)} = (1-\beta)p(z) + \beta \ . \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.3.8)$$

Then $p\epsilon[1,1]$ since$f \in \Sigma^*(\alpha)$, by differentiating (3.3.5) we obtain

Re$\psi\big(p(z), zp'(z)\big) > 0$ where $\psi(r,s) = (1-\beta)r + \beta + \frac{(1+\beta)s}{\gamma+1+\beta-(1-\beta)r}$

If $\beta \leq \beta(\alpha,\gamma)$ where $\beta(\alpha,\gamma)$ is given by (3.3.2). Using this result in (3.3.8) together with $\beta < 1$ shows that $I_{\gamma}[f](z)\epsilon\Sigma^*(\beta)$.

2.4.2. THEOREM 3.3.2

Macun, (1996) let $\alpha < 1$, $0 < \gamma, f \in \Sigma^*(\alpha)$ and $F = I_\gamma[f]$, where I_γ is defined by (3.3.1). IF $F(z) \neq 0$ for $z\epsilon\Delta$, then $I_\gamma[f](z)\epsilon\Sigma^*(\beta)$ where $\beta = \beta(\alpha,\gamma)$ is defined by (3.3.2) if $\alpha = \frac{-1}{2(\gamma+1)}$ in this theorem, then $\beta(\alpha,\gamma) = 0$ and we obtain the following corollary.

Corollary 3.3.1 let $f \in \Sigma$ with $\mathrm{Re}\left[\frac{zf''(z)}{f'(z)}\right] < \frac{1}{2(\gamma+1)} F = I_\gamma[f]$, where I_γ is defined by (3.3.1) and IF $F(z) \neq 0$ for $z\epsilon\Delta$, then $I_\gamma[f]\epsilon\Sigma^*$.

2.4.3. THEOREM 3.3.3

let $0 \leq \alpha < 1$ and $0 < \gamma \leq 1$ if $f\epsilon\Sigma_1^*(\alpha)$ then $I_\gamma[f](z)\epsilon\Sigma_1^*(\beta)$ where

$$\beta = \beta(\alpha,\gamma) = \frac{1}{2}\left[\alpha + \gamma + 2 - \sqrt{(\gamma-\alpha)^2 + 4\gamma}\right] \quad . \qquad (3.3.9)$$

2.4.4. THEOREM 3.3.

let $\alpha < 1$ and $0 < \gamma$ if $f\epsilon\Sigma_1^*(\alpha)$ and $F = I_\gamma[f]$, where I_γ is defined by (3.3.1) and IF $F(z) \neq 0$ for $z\epsilon\Delta$, then $I_\gamma[f](z)\epsilon\Sigma_1^*(\beta)$ where $\beta(\alpha,\gamma)$ is given by (5.3.9) if $\alpha = \frac{1}{(\gamma+1)}$, then $\beta(\alpha,\gamma) = 0$.

Example 2

Let $f(z) = \frac{(1+z^2)^{1-\alpha}}{z}$ with $\frac{-1}{(\gamma+1)} \leq \alpha < 1$ and $0 < \gamma \leq 2$. This function satisfies

$\frac{zf''(z)}{f'(z)} = \frac{1+(2\alpha-1)z^2}{1+z^2}$ and $f\epsilon\Sigma_1^*(\alpha)$. in this case

$$F(z) = I_\gamma[f](z) \equiv \frac{\gamma}{z^{\gamma+1}} \int_0^z f(1+t^2)^{1-\alpha} t^{\gamma-1} dt$$

From theorem(3.3.4) we deduce that $F\epsilon\Sigma_1^*(\beta)$ with $\beta = \beta(\alpha,\gamma) \geq 0$. In particular, this shows that $F(z)$ is starlike univalent

3. Conclusion

Geometric Function Theory (GFT) stands as a classical field within mathematics, yet it continually evolves, finding new applications across various disciplines including modern classical physics and beyond. Throughout this study, we have delved into several fundamental concepts within GFT, focusing particularly on analog results related to meromorphic functions and the characterization of starlikeness among these functions. Our investigation has yielded significant contributions to the existing body of literature in GFT. By exploring analog results of meromorphic functions, we have extended the understanding of their behaviors and properties, particularly in contexts where traditional analytic functions might not suffice. The study of starlikeness, which concerns the convexity of certain domains associated with meromorphic functions, has provided deeper insights into the geometric aspects of these functions.

Moreover, our findings not only extend but also refine the results documented in previous studies. Through rigorous analysis and exploration of various mathematical techniques, we have enhanced the theoretical framework surrounding GFT, thereby contributing to a more comprehensive understanding of its applications.

Looking forward, the applications of GFT continue to expand into new and diverse areas, driven by its foundational principles and the insights gained from studies such as ours. As GFT intersects with fields like physics, engineering, and computer science, its theoretical underpinnings and practical implications become increasingly relevant and valuable.

Our study underscores the enduring significance of Geometric Function Theory in mathematics and its ongoing relevance in advancing knowledge across disciplines. By deepening our understanding of analog results and starlikeness in meromorphic functions, we contribute to the rich tapestry of mathematical inquiry while paving the way for further exploration and application in both theory and practice.

Compliance with ethical standards

Disclosure of conflict of interest

The authors declares no conflict of interes.

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