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REPARAMETRIZATION INVARIANCE AND ITS IMPLICATIONS FOR PROBLEM OF TIME

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ABSTRACT

Merging quantum mechanics with general relativity remains one of the central challenges in theoretical physics, primarily manifesting as the “problem of time”. In the canonical quantization approach, general relativity yields the Wheeler-DeWitt equation, where the resulting dynamics exhibit invariance under time reparametrizations, leading to a static, time-independent description of the universe. In this paper we briefly outline the origin of the problem of time and subsequently analyze a deparametrization scheme for a system whose motion is non-linear and coupled between two variables.

Keywords: General Relativity; Quantum Gravity; Problem of Time; Deparametrization

1 INTRODUCTION

The canonical formulation of general relativity [1] treats the spacetime metric, as the fundamental dynamical variable. Upon quantization, this formulation yields the Wheeler-DeWitt equation [2]. Within this scheme, the total Hamiltonian consists of a linear combination of Hamiltonian and diffeomorphism constraints [3]. The Hamiltonian constraint forms the core of the problem of time in quantum gravity. In the quantum regime, imposing this constraint forces the quantum Hamiltonian operator onto the physical state to yield zero, implying that the wavefunction does not evolve relative to an external time parameter.

The Wheeler-DeWitt equation thus describes a state that remains functionally “frozen” in time. This behaviour stems from the inherent reparametrization invariance of general relativity, where time acts merely as a coordinate label devoid of intrinsic physical meaning [4]. In this paper, we investigate a system featuring a specific form of reparametrization invariance to examine whether the problem of time arises in this particular model and evaluate the efficacy of its deparametrization.

2 EVOLUTION AND CONSTRAINT

We consider the action [5]

$$S(\xi, \eta) = \int d\tau \sqrt{\frac{m}{2}(\dot{\xi}^2 + \dot{\eta}^2)} \left(E - \frac{1}{2} k(\xi - \eta)^2 \right), \quad (1)$$

where $\dot{\xi} = \frac{d\xi}{d\tau}$ and $\dot{\eta} = \frac{d\eta}{d\tau}$, and τ is a parameter. In this expression ξ and η are coordinates of two particles, E is a constant energy term and m is mass of each particle. This action is not strictly reparametrization invariant in standard sense; however, it can be proved that under some approximation the physics described this action is reparametrization invariant. Due this invariance the canonical Hamiltonian for this action is identically zero. A zero Hamiltonian is the property of a system where time evolution (in our case evolution with respect to τ) is not generated by an energy function. This reparametrization invariance is an indication that the dynamics of this system is governed by the constraints which arise here. In such case, one typically needs to fix a gauge to obtain non-trivial equations of motion in terms of the selected evolution parameter. This parameter might then be identified with physical time in some contexts.

Now the total Hamiltonian of the system will be

$$H_T = \lambda \left(p_\xi^2 + p_\eta^2 - \frac{m}{2} \left(E - \frac{1}{2} k(\xi - \eta)^2 \right) \right), \quad (2)$$

where λ is Lagrange multiplier, p_ξ and p_η are canonical momenta.

The primary constraint for this theory is

$$\phi(\xi, \eta, p_\xi, p_\eta) = p_\xi^2 + p_\eta^2 - \frac{m}{2} \left(E - \frac{1}{2} k(\xi - \eta)^2 \right) = 0. \quad (3)$$

Due to this constraint the dynamics is restricted to the surface in phase space where this constraint is satisfied [6]. Since our action is parametrized by an arbitrary parameter τ , and the Hamiltonian does not explicitly depend on any external time parameter, the evolution is generated by the constraint itself. Equations of motion are now derived using the H_T and Poisson brackets as:

$$\dot{\xi} = \{\xi, H_T\} = \{\xi, \lambda\phi\} = 2\lambda p_\xi, \quad (4)$$

$$\dot{\eta} = 2\lambda p_\eta, \quad (5)$$

$$\dot{p}_\xi = \frac{1}{2} \lambda m k(\xi - \eta) \quad (6)$$

and

$$\dot{p}_\eta = -\frac{1}{2} \lambda m k(\xi - \eta). \quad (7)$$

It should be remembered that Lagrange multiplier λ is related to the choice of parametrization. By choosing a gauge one can obtain specific equations of motion in terms of a chosen evolution parameter. We, therefore, observe that the dynamics so far is governed by the total Hamiltonian along with the constraint as given by equation (3).

3 THE PROBLEM OF TIME IN OUR MODEL

The problem of time appears as soon as we start quantizing the theory. According to Dirac's proposal [7] the total Hamiltonian becomes the operator on physical quantum states and gives zero. It means that evolution is completely lost and there is no dynamics on quantum level. This is also the case with our proposed model. The total Hamiltonian is a constraint and thus when applied to the quantum states, it annihilates it and there is nothing like time evolution. Therefore,

$$\widehat{H}_T \psi = 0, \quad (8)$$

where $\widehat{H}_T$ is the Hamiltonian operator and ψ is the physical quantum state of the system under consideration.

4 DEPARAMETRIZATION AND RELATED PROBLEMS

In context of our model where related quantum theory seems to be fundamentally timeless, deparametrization [8] is attempted to recover a notion of time evolution. This helps in extracting the physical degrees of freedom and their evolution. We will not attempt to give a complete mathematical analysis of this deparametrization scheme in this short communication. We, however, briefly discuss the challenges that one has to face in this procedure of deparametrization.

There are several challenges that must be solved before a successful deparametrization. We briefly discuss them one by one as follows:

- **Finding a monotonic variable-** It is not always guaranteed that globally monotonic variable exists for all physical trajectories of a system.
- **Invertibility-** The relationship between the physical time and parameter τ might not be globally invertible or might lead to multivalued functions of τ .
- **Singularities-** The clock variable might stop or reverse its direction at certain points, leading to singularities in the deparametrized equations.
- **Quantum ambiguities-** In quantum theory the choice of time variable can affect the resulting quantum dynamics and unitarity of the evolution.

For our model the deparametrization would involve choosing either ξ or η (or some function of them) as the time variable, provided their evolution is monotonic. Then we would try to express the other variables and its dynamics in terms of this chosen time. Given the coupled nature of ξ and η through $(\xi - \eta)^2$ term, finding a globally monotonic variable and successfully deparametrizing the theory might be challenging. This coupling might lead to a scenario where neither $\xi(\tau)$ nor $\eta(\tau)$ is guaranteed to be monotonic e. g. they could oscillate relative to each other. The presence of term $E - \frac{1}{2}k(\xi - \eta)^2$ under a square root sign introduces a constraint on the allowed region of (ξ, η) space (i.e. $E \geq \frac{1}{2}k(\xi - \eta)^2$). If the system reaches the boundaries of this region, the speed might become zero, potentially causing issues with using the coordinates as time variables.

5 CONCLUSION

Due to inherently coupled nature of the dynamics and the presence of oscillatory terms induced by $(\xi - \eta)^2$, the model cannot be straightforwardly deparametrized globally with respect to either ξ or η across all trajectories. Nevertheless, local deparametrization remains viable within specific regions of the phase space and for particular modes of motion, provided that one of the coordinates-or an appropriate linear combination thereof- exhibits monotonic behavior.

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