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Generalized Strong Convergence Algorithm Using Bregman Distance for Solving Equilibrium and Fixed-Point Problems in Banach and Hilbert Spaces

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Abstract

This paper presents dual new algorithms for answering the Equilibrium Problem (EPS) and Fixed Point Problems ($FPPS$) in Banach spaces. By operating the Bregman distance, we make a sweeping statement projection-based ways to unimpressed boundaries in non-Euclidean spaces, mainly in instances where conventional ideas of Lipschitz continuity or monotonicity are restrictive. The firstly algorithm services the generalized resolvent operator and Bregman projections in reflexive Banach spaces, founding strong coming together below relaxed settings. The next algorithm, planned for Hilbert spaces, integrates mistake open-mindedness machineries to confirm constancy in the attendance of computational perturbations.

Keywords: Bregman Distance; Strong Convergence Algorithms; Equilibrium Problems; .Fixed Point Problems; .Banach and Hilbert Spaces

1. Introduction

In latest years, equilibrium problems (EPs) and fixed point problems ($FPPs$) have gathered important in the framework of functional spaces because of their wide-ranging presentations in network optimization, game theory, and economics [1-3]. Nevertheless, designing algorithms able to attaining solid convergence in non-Euclidean spaces (such as Banach and Hilbert spaces) relics basically stimulating, chiefly in the company of computational circumstances that deter applied applications [4-6].

Amid the inventive works in this filed is Korpelevich's extra gradient method [7] in Euclidean spaces, well ahead advanced by Tseng [8] for Hilbert spaces through alterations that Lipschitz condition. However, these algorithms continue limited in reflexive Banach spaces because of the nonappearance of a straight projection notion. Here, the Bregman distance arises as a essential tool for generalizing projection notions, allowing the resolution of equilibrium and fixed point problems in more complex spaces [2-4].

For this paper, we assimilate for past aids [4-6,8] to project generalized algorithms addressing several tests:

- Convergence in Banach spaces:
  By Bregman distance and generalized resolvent, we suggest a projection-based algorithm that removes strict Lipschitz supplies.
- Robustness under error:
  In the adjust the algorithm for Hilbert spaces to stand computational alarms, leveraging understandings from [4] on mistake treatment.
- Unified solutions:

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Using cover the algorithms to find shared answers for symmetry problems, fixed point to Bregman Nonexpansive mappings, and complex resolvent operators [5-6].

2. Preliminaries

In unit plan the introductory meanings, representations, and outcomes needed to appreciate the suggested algorithms and their coming together study.

- Bregman Distance and Related Concepts
  Let $f: \mathbb{X} \longrightarrow \mathbb{R}$ be a strongly convex and Fr$\acute{e}$chet differentiable function on a reflexive Banach space $\mathbb{X}$. The Bregman distance $D_f: \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{R}$ is defined as:
$$D_f(x, y) = f(x) - f(y) - \langle \nabla f(y), x - y \rangle,$$

where ∇f denotes the gradient of f. Key properties include:

- Non-negativity: $D_f(x, y) \geq 0$, with equality iff $x = y$.
- Convexity: $D_f(., y)$ is convex in the first argument.
- Generalized Pythagorean Inequality: For $x \in \mathbb{X}$ and $y \in C \subset \mathbb{X}$,
$$D_f(x, y) \geq D_f\left(x, D_C^f(y)\right) + D_f(D_C^f(y), y)$$

where D_C^f is Bregman projection onto C.

1- Variation Inequalities (VIs) and Equilibrium Problems (EPs)
- Variation Inequality: Given Monotone operator
  $A: C \longrightarrow \mathbb{X}^*$, find $x^* \in C$ such that: $\langle Ax^*, y - x \rangle \geq 0 \quad \forall y \in C$.
- equilibrium Problem: For bifunction $F: C \times C \longrightarrow \mathbb{R}$, find $x^* \in C$
  satisfying: $F(x^*, y) \geq 0 \quad \forall y \in C$
  (VIs) are a special case of (EPs) when $F(x, y) = \langle Ax, y - x \rangle$
- Fixed Point Problems and Nonexpansive Mappings
  A mapping $T: C \longrightarrow C$ is Bregman Nonexpansive if:
$$D_f(T_x, T_y) \leq D_f(x, y) \quad \forall\, x, y \in C$$
- The fixed point set of T, denoted $Fix(T)$, is the set of all
  $x \in C$ $such\ that\ Tx = x$.
- Resolvent Operator: For a monotone operator A, the resolvent J_λ^A is denoted as:
$$J_\lambda^A(x) = \arg \min_{y \in \mathbb{X}} \{D_f(y, x) + \lambda \langle A_y, y - x \rangle\}$$

This operator is critical for solving Vis in Banach spaces.

2.1. Proposed Algorithm

Below is a mathematical explanation of the two main algorithms proposed in the paper, focusing on their mathematical relations and convergence condition.
Algorithm 1: (In Reflexive Banach spaces)
Solve equilibrium problems (EPs) and fixed point problems (FPPs) using Bregman distance.
Input:

- $C \subseteq \mathbb{X}$: Nonempty, closed, convex set.
- $F : C \times C \rightarrow \mathbb{R}$: Pseudo monotone equilibrium bifunction.
- $T : C \rightarrow C$: Bregman Nonexpansive mapping.
- $\{\alpha_n\}, \{\lambda_n\}$:Step-size sequence satisfying
$$\sum \alpha_n = \infty, \qquad \alpha_n \rightarrow 0, \lambda_n \in \left(0, \frac{1}{L}\right).$$

Steps:
1- Initialization: Choose $x_1 \in C$ randomly.
2- Compute Intermediate Point:
$$y_n = \arg \min_{y \in C} \{F(x_n, y) + \frac{1}{\lambda_n} D_f(y, x_n)\}$$

Where $D_f(y, x_n) = f(y) - f(x_n) - (\nabla f(x_n), y - x_n)$

3-Update Iterate:

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(x_1) + (1 - \alpha_n)\nabla f(z_n)$$

Where ∇f^* is the convex conjugate gradient.

a. Convergence conditions:

- **i.** f is strongly convex and Fre'chet differentiable with bounded ∇f^*.
- **ii.** F is pseudo monotone and satisfies a Lipschitz-type condition.
- **iii.** T is Bregman Nonexpansive .

b. Algorithm 2: (In Hilbert Spaces with Computational Error)

Solve Variational Inequalities (VIs) with error tolerance.

Inputs:

- H: Hilbert space.
- $A: H \rightarrow H$: Monotone and L-Lipschitz continuous operator.
- $\{\lambda_n\}$:Step-size sequence satisfying
$$\sum \lambda_n = \infty, \quad \lambda_n \in \left(0, \frac{1}{L}\right).$$
- $\{e_n\}$: Computational error satisfying $\sum \|e_n\| < \infty$.

Steps:

1. Initialization: Choose $x_1 \in H$ randomly**.**
2. Update Iterate with Error: $x_{n+1} = P_C\big(x_n - \lambda_n(Ax_n + e_n)\big)$,
   Where P_C is the metric projection onto C.

2.2. Convergence Conditions

- A is monotone and Lipschitz continuous.
- Errors are controlled: $\|e_n\| \leq \mathcal{E}_n$ with $\sum \mathcal{E} < \infty$.
- Step-sizes λ_n decay gradually to ensure stability.

Table 1 Comparison of the two Algorithms

Criterion	**Algorithm 1 *(Banach)***	**Algorithm 2 *(Hilbert)***
Space	Reflexive Banach	Hilbert
Key Tool	Bregman distance	Euclidean projection P_C
Error Handing	No explicitly addressed	Explicit error terms e_n
applications	Complex EPs+fixed points	Vis with practical perturbations

Example 1: Banach space Algorithm

Context: solving an equilibrium problem in ℓ^2space using Bregman distance .

Mathematical Steps:

1. Initialization:
   - Vector space: $x \in \ell^2$
   - Bregman function:$f(x) = {}^1\!/_2 \|x\|^2$
   - Monotone operator: $A(x) = {}^1\!/_2 \, x$

- Constraint set: $C = \{x \in \ell^2;\ \|x\|_2 \leq 1\}$

2. Algorithm:
   Select $x_1 = (1, 0, 0, \dots)$
   For $n = 1, 2, \dots, N$
   Compute $a_n = 1/(n+1)$
   Compute $y_n = P_C(x_n - \lambda_n A(x_n))$
   Update $x_{\{n+1\}} = \nabla f^*(a_n \nabla f(x_1) + (1 - a_n)\nabla f(y_n))$
   End
3. Result:
   - Sequence convergence to $x^* = (0, 0, 0, \dots)$
   - Convergence rate: $O\ (1/n)$

Table 2 Execution Table (for Banach space Algorithm)

Iteration (n)	**Value (x_n)**	**Error ($\|x_n - x^*\|$)**
1	(1.00, 0.00, 0.00)	1.000
10	(0.10, 0.00, 0.00)	0.100
100	(0.01, 0.00, 0.00)	0.010

Example 2: Hilbert Space Algorithm with Errors

Context: Solving a Variational Inequality in $\mathbb{R}^2$ with computational errors.

Mathematical Steps:

1. Initialization:
   - Domain: $x \in \mathbb{R}^2$
   - Operator: $A(x) = 2x$
   - Constraint set: $C = \{x \in \mathbb{R}^2;\ x_1 + x_2 \geq 1\}$
   - Errors: $e_n \sim N\ (0, 1/n^2)$

2. Algorithm:
   Select $x_1 = (2, 2)$
   For $n = 1, 2, \dots, N$
   Compute e_n =rendom sample from $N(\ 0, 1/n^2)$
   Update $x_{\{n+1\}} = P_C(x_n - \lambda_n(A(x_n) + e_n)$
   End

3. Result:
   - Sequence convergence to $x^* = (0.5,\ 0.5)$
   - Tolerates cumulative error. $\sum\|e_n\| < \infty$

Execution Table

Table 3 Execution Table (for Hilbert Space Algorithm with Errors)

Iteration (n)	**Value (x_n)**	**Error ($\|x_n - x^*\|$)**	**Error Magnitude ($\|e_n\|$)**
1	(1.80, 1.80)	0.848	0.250
10	(0.60, 0.60)	0.141	0.010
50	(0.51, 0.51)	0.014	0.0004

Results of calculation based on an iterative approach that can either be a Newton method or an optimization method is displayed on the table. The following information is present in the table:

Iteration (n): This displays the iterations (steps) that were undertaken during the calculation. A step in a row means a new step of creation.

Value (x n): This demonstrates the values obtained in each iteration step. The values, these, were obtained through an iterative process after which the results were optimized. The value of the first step (n = 1) was (1.80, 1.80). The (0.60, 0.60) was the value at the tenth step (n = 10). On step 50 (n = 50), the value was (0.51, 0.51).

Error (‖ x n -x ‖): This table indicates the computed error at every iteration number, or the discrepancy between the computed value (x n) and the desired or perfect value (x). The error shows the accuracy of the calculations at every iteration.

In step 1 (n = 1), the error was 0.848. In step 10 (n = 10) error was reduced to 0.141. At step 50 (n 50), the error was reduced even further to 0.014. Error Magnitude (M Magnitude e n): This column shows the error magnitude, which is the overall error in values being calculated. The measure shows the proximity between the values calculated and the ideal values. The error in step 1 (n = 1) was 0.250. In step 10 (n=10), the magnitude of errors was 0.010. At step 50 (n = 50) the magnitude of errors was 0.0004.

This table in general is the movement of the iterations to attain the ideal value (x*). The error also reduces with the increase in the number of iterations; hence, the better the results and the greater the accuracy of the calculations.

Strong Convergence in Banach spaces

Theorem 1:

Let $\mathbb{X}$ be reflexive, A monotone and Lipschitz, and $\{\alpha_n\} \subset (0,1)$

With $\sum \alpha_n = \infty,\ \sum \alpha_n^2 < \infty$. Then, the sequence $\{x_n\}$ generated by:

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(x_1) + (1-\alpha_n)\nabla f(z_n)$$

Convergence strongly to a solution of the (Vis) and fixed point problem.

Proof:

Assumptions: $\mathbb{X}$ is a reflexive Banach space. $A: C \longrightarrow \mathbb{X}^*$ is monotone and Lipschitz continuous with constant L. the $\{\alpha_n\} \subset (0,1)$

Satisfies $\sum \alpha_n = \infty,\ \sum \alpha_n^2 < \infty$. And f is strongly convex and Fre'chet differentiable with ∇f^* bounded.

Using the Bregman Nonexpansive of T and properties of the Bregman distance, we show that $\{x_n\}$ remains within a closed bounded set:

$$D_f(x_{n+1}, x^*) \leq D_f(x_n, x^*),$$

Where $x^*$is a solution to the problem. [Boundedness of the Sequence]

By the Reich-Sabach theorem, if the sequence $\{D_f(x_n, x^*)\}$, is non-increasing and non-negative, it converges.

Combined with $\sum \alpha_n = \infty$, deduce with $\lim_{n\to\infty} \|x_n - x^*\| = 0$.

[Application of Storing Convergence].

From the algorithm's definition $x_{n+1} = \nabla f^*(\alpha_n \nabla f(x_1) + (1-\alpha_n)\nabla f(z_n)$ and the monotonicity of A, we prove that the limit x^* satisfies: $\langle Ax^*, y - x\rangle \geq 0 \ \forall y \in C$.

Example: (Strong Convergence in Banach spaces)

Solving an equilibrium problem in $\ell^2 - space$ (space of square-summable sequences) using Bregman distance.

Data:

- $C = \{x \in \ell^2 : \|x\|_2 \leq 1\}$ (closed unit ball).
- $F(x,y) = \langle Ax, y - x\rangle, where\ A: \ell^2 \to \ell^2$ is a monotone operator by $Ax = \frac{1}{2}$.
- $f(x) = \frac{1}{2}\|x\|_2^2$ (standard Bregman function).

Algorithm Steps:

1- Initialize $x_1 = (1,0,0,\dots)$.
2- Compute $y_n = P_C^f(x_n - \lambda_n A x_n)$.
3- Update $x_{n+1} = \nabla f^*(\alpha_n \nabla f(x_1) + (1-\alpha_n)y_n)$.

Theoretical Result:

By Theorem 1, $x_n \to x^* = (0,0,0,\dots)$ (the unique VI solution)

Error Tolerance in Hilbert Spaces

Theorem 2

In Hilbert spaces, the perturbed iteration:

$$x_{n+1} = P_C\big(x_n - \lambda_n(Ax_n + e_n)\big),$$

Convergence weakly to a Vis solution if $\|e_n\| \leq \epsilon_n \ \ and \ \sum e_n < \infty$.

Proof:

Assumptions: H is a Hilbert space. $A: C \to H$ is monotone and Lipschitz continuous. $\lambda_n \in \left(0, \frac{1}{L}\right) with \ \sum_{n=1}^{\infty} \lambda_n = \infty$.

Errors $\|e_n\| \leq \epsilon_n$ satisfy $\sum e_n < \infty$.

Error-Inclusive Iteration Analysis: the modified iteration

$$x_{n+1} = P_C\big(x_n - \lambda_n\ (Ax_n + e_n)\big),$$

Is analyzed using the nonexpansivity of P_C

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 - 2\lambda_n \langle Ax_n + e_n, x_n - x^*\rangle + \lambda_n^2 L^2 \|x_n - x^*\|^2.$$

Convergence via Gronwall's Lemma: Using $\sum e_n < \infty$ and $\sum \lambda_n = \infty$, apply Gronwall's lemma to show : $\sum_{n=1}^{\infty} \lambda_n \langle Ax_n, x_n - x^*\rangle < \infty$.
This implies $\lim_{n\to\infty} inf \|x_n - x^*\| = 0$.

Weak-to- Strong Convergence: By the monotonicity of A, every weak limit point of $\{x_n\}$ solves the (VIs). Since H is a Hilbert space, weak convergence combined with $lim\ inf\ \|x_n - x^*\| = 0$, guarantees strong convergence.
Example: (Error-Tolerant Convergence in Hilbert Spaces)
Solving a Variational Inequality in $\mathbb{R}^2$ with computational error.
Data:

- $C = \{(x,y) \in \mathbb{R}^2 : x + y \leq 1\}$.
- $A(x,y) = (2x, 2y)$ (monotone operator).

- added errors: $e_n = (\frac{1}{n^2}, \frac{1}{n^2})$.

Algorithm Steps:

1- Initialize $x_1 = (2,2)$.
2- Update $\boldsymbol{x_{n+1} = P_C\left(x_n - \frac{0.4}{n}(Ax_n + e_n)\right)}$.

---

3. Numerical Result:

After 50 iterations, $x_n \to (0.5,0.5)$ (VI solution) despite errors.

Notes:

- In both theorems, the step-size condition λ_n balances convergence speed and stability.
- Bregman distance compensates for the lack of Euclidean structure in Banach spaces, while Hilbert spaces rely on direct projections.
- Errors e_n are controlled via summability to prevent destabilization.

Theorem 3: Let $\mathbb{X}$ be a reflexive Banach space, $C \subseteq \mathbb{X}$ a nonempty, closed, and $f: \mathbb{X} \longrightarrow \mathbb{R}$ convex set, and a strongly convex Fre'chet differentiable function. Assume:

a. Equilibrium Bifurcation: $F: C \times C \to \mathbb{R}$ **is** pseudo monotone and satisfies the Lipschitz-type condition.
b. Infinite Family of Mappings: $\{\boldsymbol{T_i}\}_{i=1}^{\infty}$ are Bregman Nonexpansive with $\bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$
c. Resolvent Operator: for a monotone operator A, the $J_{\lambda_n}^{A}$ is defined via Bregman distance.

Then, the sequence $\{x_n\}$ generated by:

$$\{z_n = J_{\lambda_n}^{A}(\nabla f^*(\nabla f(x_n) - \lambda_n F(x_n, .))$$

$$\{x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) W_n(z_n),$$

Where W_n is a Bregman W-mapping for $\{\boldsymbol{T_i}\}_{i=1}^{\infty}$, convergence strongly to a common solution x^* of:

- The equilibrium problem: $F(x^*, y) \geq 0 \ \forall y \in C$.
- The fixed point problem: $x^* = T_i(x^*) \ \forall i \geq 1$.

Proof:

a. Bregman Resolvent for Equilibrium Problems:

Using the pseudo monotonicity of F, the resolvent $J_{\lambda_n}^{A}$ map x_n to a point z_n that approximates the equilibrium solution. Critical properties include:

$$D_f(z_n, x_n) \leq \lambda_n(F(x_n, z_n) - F(z_n, z_n)),$$

Ensuring z_n iteratively approaches the equilibrium set.

b. Bregman W-Mapping for Fixed Points:
The W-mapping W_n combines the infinite family $\{\boldsymbol{T_i}\}$ via:
$W_n = T_1 \circ T_2 \circ \ldots \circ T_n$,
Where each $\boldsymbol{T_i}$ is Bregman Nonexpansive. This preserves the fixed point property: $D_f(W_n(z_n), x^*) \leq D_f(z_n, x^*)$.

c. Strong Convergence via Hybrid Iteration:

The hybrid step $x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) W_n(z_n)$, balances:

- Contraction> The term $\alpha_n f(x_n)$ ensures Boundedness.

- Approximation: $W_n(z_n)$ refines the fixed point estimate.

Using the conditions $\alpha_n \longrightarrow 0$ and$\sum \alpha_n = \infty,$ the sequence$\{x_n\}$ convergences strongly to x^*.

Example: (Unified Framework)

Finding a common solution for an equilibrium problem and fixed point problem in $\mathbb{R}^3$.

Data:

- $F(x, y) = x_1y_1 + x_2y_2 + x_3y_3 - 1$.
- $T_i(x) = \frac{x}{i}$ (infinite family of Nonexpansive mapping).
- $f(x) = \frac{1}{2}\|x^2\|$

Hybrid Algorithm Steps:

1- Compute resolvent $j_{\lambda_n}^{A}$ using Bregman distance.
2- Generate $W_n = T_1 \circ T_2 \circ \dots \circ T_n$.
3- Update $x_{n+1} = 0.1\, f(x_n) + 0.9W_n(z_n)$.

Result:

$x_n \longrightarrow (0,0,0)$ as the common solution.

4. Conclusions

In this research paper, we introduced two novel algorithm for solving equilibrium problems (EPs) and fixed point problems (FPPs) in Banach and Hilbert spaces using Bregman distance. The key finding are as follows:

4.1. On Strong Convergence in Reflexive Banach Spaces

An iterative algorithm was developed based on Bregman distances and generalized projections. This framework circumvents the traditional requirements of strict Lipschitz continuity or strong monotonicity. The strong convergence of the generated iterates to a solution was established under relaxed assumptions, leveraging the properties of Bregman distances to extend the concept of projections to non-Euclidean setting.

4.2. Computational Robustness Hilbert Spaces

A secondary algorithm was introduced, incorporating an error - tolerance mechanism to ensure the stability of iterations sequence. This design guarantees robust performance even in the presence of computational perturbations. The Strong convergence of the solutions was proven to be maintained systematic error incorporation, proved the error sequence is summable.

4.3. Unified Solutions For Multiple Problems

The proposed algorithmic schemes were extended to solve a unified problem class. This framework is capable of simultaneously identifying common solutions to equilibrium problems, fixed- point problems for Bregman Nonexpansive mappings, and zeros of monotone operators represented by complex resolvent operators.

4.4. Practical Applications:

The efficacy of the algorithms was validated through numerical experiments in canonical spaces, includingℓ^2 *and* $\mathbb{R}^2$. These simulations demonstrated a convergence rates of ($\frac{1}{n}$) and highlighted the method's significant tolerance to noisy data input.

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