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η – Dual of generalized triple sequence space

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Abstract

In this paper, we introduce bounded, convergent, null and eventually alternating triple sequence spaces, i.e.,3_{l_∞},3_c, 3_{c_0} and 3_σ respectively. Then we find η -dual of order r (0< $r \leq 1$) of these spaces. Further, we check whether they are perfect or not.

Keywords: Sequence spaces; Kothe Toeplitz duals; $\alpha - duals$; $\eta - duals$.

1. Introduction

Let ω denote linear space of all sequences and X is any subset of ω .Then Kothe –

Toeplitz duals of X or α-dual of X is defined in [8] as

$$X^\alpha = \{ (a_n) \in \omega : \sum_n |a_n x_n| < \infty \text{ for all } (x_n) \in X\}.$$

Kothe –Toeplitz [1, 3, 4, 6, 10] gives idea of dual sequence space whose main results are with α-dual. Chandra and Tripathi [10] have generalized the notation of η-duals of order r> 1 . Later, Ansari and Gupta [2] worked on it and generalized the notation of Kothe and Toeplitz duals of sequence spaces by introducing the concept of η-duals of order 0< $r \leq 1$.

Let N denote the set of natural numbers. A triple sequence of complex numbers is a function x: $N \times N \times N \xrightarrow{yields}$C. We denotes triple sequence by (x_{mnp}). In this paper, sum without limits stand from 1 to ∞. Let 3_ωdenote space of all triple sequence. Then we define the spaces 3_{l_∞},3_{l_r},$3_c$and 3_{c_0} as

$$3_{l_r} = \{(a_{mnp} \in 3_\omega : \sum_m \sum_n \sum_p |a_{mnp}|^r < \infty\};$$

$$3_{l_\infty} = \{(a_{mnp}) \in 3_\omega : sup_{m,n,p} |a_{mnp}| < \infty\};$$

$$3_c = \{(a_{mnp}) \in 3_\omega : a_{mnp} \to l \; as \min(m,n,p) \to \infty \;\; for \; some \; l \in C\};$$

$$3_{c_0} = \{(a_{mnp}) \in 3_\omega : a_{mnp} \to 0 \; as \; min \; (m,n,p) \to \infty\}.$$

Clearly, from the above expression, we have

$$3_{c_0} \subseteq 3_c \subseteq 3_{c\infty}$$

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Let E be any non empty subset of $3_{\omega.}$Then we define η-dual of E of order r, for

$$0 < r \leq 1 \; as$$

$$E^{\eta}=\{(a_{mnp}) \in 3_{\omega}:\sum_m \sum_n \sum_p \left|a_{mnp}x_{mnp}\right|^r < \infty\},$$

Where $(x_{mnp}) \in E$. Any non empty subset E of $3_{\omega}$is said to be perfect if $E^{\eta\eta}$=E.

2. Results

2.1. Lemma 2.1

The following statements hold

- E^{η} is a linear subspace of 3_{ω} for every E⊆ 3_{ω}
- If E⊆ F, then $F^{\eta} \subseteq E^{\eta}$.
- E⊆ $E^{\eta\eta}$

2.1.1. Theorem 2.1

$(\mathbf{3}_{l_r})^{\boldsymbol{\eta}}$ = $3_{l_\infty}$and$(\mathbf{3}_{l_\infty})^{\boldsymbol{\eta}}$ = 3_{l_r} .The spaces $3_{l_\infty}$and $3_{l_r}$are perfect for 1≥ $r > 0$.

Proof

Let $(a_{mnp}) \in 3_{l_\infty}$be any element .Then, we have

$$sup_{m,n,p}\left|a_{mnp}\right| < \infty\}. \ldots\ldots\ldots\ldots\ldots\ldots. (2.1)$$

If $(\, x_{mnp}) \in 3_{l_r}$ is an arbitrary element, then

$$\sum_m \sum_n \sum_p \left|x_{mnp}\right|^r < \infty. \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.2)$$

Now consider

$$\sum_m \sum_n \sum_p \left|a_{mnp}x_{mnp}\right|^r \leq sup_{m,n,p}\left|a_{mnp}\right|^r \sum_m \sum_n \sum_p \left|x_{mnp}\right|^r < \infty.$$

this impies $(a_{mnp}) \in (3_{l_r})^{\eta}$. Hence

$$3_{l_\infty} \subseteq (3_{l_r})^{\eta} \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (2.3)$$

For the converse part of the theorem, if $(a_{mnp}) \notin 3_{\mathrm{l}_\infty}$then ∃ a subsequence say (a_{iipi}) of (a_{mnp}) such that

$$a_{iipi} \geq i^s \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2.4)$$

For some s> 0 *where* $sr > 1$.

Now we define a sequence (x_{mnp})as

$$x_{mnp}=\begin{cases}\frac{1}{i^s} & m=i=n, p=p_i \in N\\ 0 & ,otherwise\end{cases}$$

Then $$\sum_m \sum_n \sum_p |x_{mnp}|^r = \sum_i (\frac{1}{i^s})^r < \infty$$

by using equation (2.4).

$$\text{But } \sum_m \sum_n \sum_p |a_{mnp} x_{mnp}|^r = \sum_m \sum_n \sum_p |a_{mnp}|^r \, |x_{mnp}|^r$$

$$\geq \sum_i \left| i^s \times \frac{1}{i^s} \right| \to \infty$$

$$\Longrightarrow (a_{mnp}) \notin (3_{l_r})^\eta$$

$$\Longrightarrow (3_{l_r})^\eta \subseteq 3_{l_\infty} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.5)$$

From (2.3) and (2.5)

$$(3_{l_r})^\eta = 3_{l_\infty}$$

Similarly, we can prove that

$$(3_{l_\infty})^\eta = 3_{l_r}$$

Again

$$(3_{l_\infty})^{\eta\eta} = ((3_{l_\infty})^\eta)^\eta$$

$$= (3_{l_r})^\eta$$

$$= 3_{l_\infty}$$

This implies that 3_{l_∞} is perfect. Similarly, we can prove that 3_{l_r} is also perfect.

2.1.2. Theorem2.2

$(3_{c_0})^\eta = (3_c)^\eta = 3_{l_r}$ and both the spaces $3_{c_0,} 3_c$ are not perfect for $1 \geq r > 0$.

Proof

By the definition of $3_{c_0,}$ *and* 3_{l_∞}, we have $3_{c_0,} \subseteq 3_{l_\infty}$.Taking η −dual of

Both sides and using Lemma 2.1, we get

$$(3_{l_\infty})^\eta \subseteq (3_{c_0})^\eta$$

$$\Longrightarrow 3_{l_r} \subseteq (3_{c_0})^\eta.$$

Again, let$(a_{mnp}) \in$ $(3_{c_0})^\eta$ be an arbitrary element.Then

$$\sum_m \sum_n \sum_p \left|a_{mnp}x_{mnp}\right|^r < \infty, for\ (x_{mnp}) \in 3_{c_0}$$

$$\Longrightarrow \sum_m \sum_n \sum_p \left|a_{mnp}\right|^r \left|x_{mnp}\right|^r < \infty$$

$$\Rightarrow \sum_m \sum_n \sum_p \left|(a_{mnp})^r\right|\left|z_{mnp}\right| < \infty, \text{Where } (z_{mnp})=(x_{mnp})^r$$

$$\Rightarrow (a_{mnp})^r \in (3_{c_0})^\alpha = 3_{l_1}$$

$$\Rightarrow (a_{mnp}) \in 3_{\mathrm{l_r}}, \text{ for all} (a_{mnp}) \in (3_{c_0})^\eta$$

$$\text{So, } (3_{c_0})^\eta \subseteq 3_{\mathrm{l_r}}$$

Hence

$$(3_{c_0})^\eta = 3_{\mathrm{l_r}} \ldots\ldots\ldots\ldots\ldots\ldots(2.6)$$

Further

$$3_{c_0} \subseteq 3_c$$

$$(3_c)^\eta \subseteq (3_{c_0})^\eta \text{ by Lemma 2.1}$$

$$(3_c)^\eta \subseteq 3_{\mathrm{l_r}}$$

Also,

$$3_c \subseteq 3_{\mathrm{l_\infty}}$$

$$\Rightarrow (3_{l_\infty})^\eta \subseteq (3_c)^\eta \text{By Lemma 2.1}$$

$$\Rightarrow 3_{\mathrm{l_r}} \subseteq (3_c)^\eta$$

So,

$$(3_c)^\eta = 3_{\mathrm{l_\infty}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.7)$$

For perfectness, let us consider

$$(3_{c_0})^{\eta\eta} = (3_{l_r})^\eta$$

$$= 3_{\mathrm{l_\infty}}$$

$$\text{i.e. } (3_{c_0})^{\eta\eta} \neq 3_{c_0}$$

This implies that 3_{c_0} is not perfect. Similarly, we can prove that 3_c is also not perfect.

2.2. Definition2.3

The space 3_σ of all eventually alternating triple sequence space is defined as

$3_\sigma = \{(a_{mnp}) \in 3_\omega : a_{mnp} = -a_{m,n,p+1} = -a_{m,n+1,p} = -a_{m,n+1,p}$ for any m, n, p$\geq m_0, n_0, p_0\}$.

2.2.1. Theorem 2.4

$(\mathbf{3}_\sigma)^{\boldsymbol{\eta}} = 3_{\mathrm{l_r}}$ and the space 3_σ is not perfect.

Proof

By the definition of 3_{σ}, we have

$$3_{\sigma} \subseteq 3_{l_{\infty}}$$

$$\Rightarrow (3_{l_{\infty}})^{\eta} \subseteq (\mathbf{3}_{\sigma})^{\boldsymbol{\eta}} \text{, by Lemma}$$

$$\Rightarrow 3_{l_r} \subseteq (\mathbf{3}_{\sigma})^{\boldsymbol{\eta}}$$

Let $(x_{mnp}) \in (\mathbf{3}_{\sigma})^{\boldsymbol{\eta}}$ be any element then

$$\sum_m \sum_n \sum_p |a_{mnp} x_{mnp}|^r < \infty, for\ all\ (a_{mnp}) \in 3_{\sigma}. \ldots\ldots\ldots\ldots (2.8)$$

Let us define a sequence(a_{mnp}), as a_{mnp}= -$a_{m+1,n,p}$ = -$a_{m,n+1,p}$ =1. Then $(a_{mnp}) \in 3_{\sigma}$

Using this in eqs. (2.8), we get

$$\sum_m \sum_n \sum_p |x_{mnp}|^r < \infty$$

$$\Rightarrow (x_{mnp}) \in 3_{l_r}$$

$$\text{i.e.} \quad (\mathbf{3}_{\sigma})^{\boldsymbol{\eta}} \subseteq 3_{l_r}$$

So,

$$(\mathbf{3}_{\sigma})^{\boldsymbol{\eta}} = 3_{l_r}.$$

Again,

$$(\mathbf{3}_{\sigma})^{\boldsymbol{\eta\eta}} = (3_{l_r})^{\eta}$$

$$= 3_{l_{\infty}}.$$

Hence, $3_{\sigma}$is not perfect.

Corollary 2.1. $(3_{c_0} \cap 3_c)^{\eta}$=$3_{l_r}$, $(3_{c_0} \cup 3_c)^{\eta}$ =3_{l_r} and both spaces are not perfect .

Corollary 2.2.$(3_c \cup 3_{l_{\infty}})^{\eta}$=$3_{l_r}$,$(3_c \cap 3_{l_{\infty}})^{\eta}$=$3_{l_r}$.

3. Conclusion

The sequence spaces 3_{l_r},$3_{l_{\infty}}$ are perfect and also they are η −duals of each other. On the other hand, the sequence spaces 3_c, $3_{c_0}$and $\mathbf{3}_{\sigma}$ are not perfect.

Compliance with ethical standards

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The authors have no conflicts of interest to declare that are relevant to the content of this article .

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